Multivariate Tail risk measures for the multivariate Pareto Distributions

By Liali Hassan

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE MASTER'S DEGREE

University of Haifa
Faculty of Social Sciences
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Contents

Abstract

In this thesis we consider the multivariate tail conditional measures for multivariate Pareo type II distribution. Firstly, we derive an explicit closed-form expression for the multivariate tail conditional expectation (MTCE) for the multivariate Parto type II distribution. Secondly, we consider a multivariate tail covariance (MTCov) measure, which is a matrix- valued risk measure designed to explore the tail dispersion of multivariate loss distribution. The MTCov which is also defined for the set of different quantile levels, allows us to investigate more deeply the tail of multivariate distributions, since it focuses on the variance-covariance dependence structure of dependent risks. We also consider the multivariate tail correlation matrix (MTCorr), in order to get the correlation between any two risks. The results are illustrated with examples of multivariate Pareto.

Chapter 1

Introduction

Insurance companies set aside amount of capital from which they use as a reserve to ensure their ability to meet all future claims. The subject of determination of these amounts has been of active interest to researchers, regulators of financial products and financial institutions themselves.

Suppose that an insurance company faces the risk of losing a quantity X for some fixed period of time. This may refer to the total claims for the insurance company or to the total loss in a portfolio of investment for an individual or institution. We denote it's distribution function by $F_X(x) = P(X \leq x)$, and it's survival function by $\overline{F}_X(x) = P(X > x)$.

The concept of value at risk (VaR) has become the standard risk measure used to evaluate risk. The VaR is the amount of capital required to ensure, with high degree of certainty, that the enterprise does not become technically insolvent.
The need for VaR from the past few decades tremendous volatility in exchange rates, interest rates, and commodity prices and it's proliferation of derivative instruments for managing the risks of changes in market rates and prices. Increased trading of cash instruments and securities and the growth of financing opportunities accompanied the proliferation of derivatives. As a result, many companies have portfolios that include large numbers of (sometimes complex) cash and derivative instruments. Moreover, the magnitudes of the risks in companies portfolios often are not obvious. The result is increasing demand for a portfolio-level quantitative measure of market risk.
VaR is single, summary statistical measure of possible portfolio losses.

The promotion of VaR has prompted the study of risk measures by numerous authors. Over recent years the so-called *tail conditional expectation* (TCE) or TailVaR risk measure has become more and more popular among actuaries, because of several attractive properties. In particular, Artzner et al. (1999) demonstrated that the tail conditional expectation satisfies all requirements for a coherent risk measure.When compared to the traditional value-at-risk (VAR) measure, the tail conditional expectation provides a more conservative measure of risk for the same level of degree of confidence (1-q). This risk measure is the main subject of this thesis.

Tail conditional expectation of X is defined as

$$TCE_q(X) = E(X|X > x_q), \tag{1.1}$$

and it can be interpreted as the average of worse losses as it gives the mean amount of the tail of the distribution. This tail is usually based on the q-th quantile, the value at risk ($VaR_q(X)$) of the loss distribution with the property

$$\overline{F}_X(x_q) = 1 - q,$$

where $0 < q < 1$.

Tail conditional expectation (TCE) is regarded as one of the most important risk measures, which is incorporated into Basel II (see Fu and Jang. (2008)) and Solvency II (see Devolder and Lebegue. (2016)). The TCE risk measure was investigated by many authors and the following is a partial list of related references (Artzner (1999) , Panjer (2002), Landsman and Valdez (2003) , Landsman and Valdez (2006) , Cai and Haijun (2005) , Cai et al. (2015), Chen et al. (2014) , Katsuki and Matsumoto , Klugman et al. (2012), Ogryczak (2014)). The TCE like other risk measures, is generally defined as a mapping from a set of values of random variables to the real line, and its main goal is to quantify a financial risk by evaluating expected extreme losses.

TCE is a conditional expectation and hence, does not include information about deviation of the risk from it's expectation in the upper tail. In order to overcome this problem, Furman and Landsman (2006) introduced new risk measure, the conditional tail variance (TV). Tail variance is a measure of variability on the right tail $X > x_q$, and it is merely the conditional variance of the risk X. The tail variance is defined as

$$TV_q(X) = Var(X|X > x_q) = E((X - TCE_q(X))^2|X > x_q). \tag{1.2}$$

In addition, in this research study we consider also the multivariate tail conditional expectation (MTCE) risk measure for multivariate Pareto type II distribution. The motivation behind taking the multivariate TCE comes from the fact that unlike the traditional tail conditional expectation, the MTCE measure takes into account the covariation between dependent risks, which is the case when we are dealing with real data of losses.

The multivariate tail conditional expectation (MTCE) was introduced in Landsman et al.,2016. In 2018 the MTCE was introduced in another form, which allowing for quantile levels to obtain the different values corresponding to each risk (Landsman, Makov and Shushi, 2018).
Define $\mathbf{X} = (X_1, ..., X_n)^T$ an $n \times 1$ vector of random risks that can be dependent on each other. The MTCE of vector X is defined as

$$MTCE_{\mathbf{q}}(\mathbf{X}) = E(\mathbf{X}|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) = E(\mathbf{X}|X_1 > VaR_{q_1}(X_1), ..., X_n > VaR_{q_n}(X_n)),$$
$$\mathbf{q} = (q_1, q_2, ..., q_n) \in (0, 1)^n.$$

Where,

$VaR_{\mathbf{q}}(\mathbf{X})$ is n $\times$ 1 vector $VaR_{\mathbf{q}}(\mathbf{X}) = (VaR_{q_1}(X_1), ..., VaR_{q_n}(X_n))^T$.

The MTCE multivariate risk measure has several advantages comparing with TCE risk:
(1) It provides the expectation of the elements of collective of risks-business lines, when all business lines exceed the specified VaR-level threshold.
(2) The calculation of the proposed measure is relatively simpler than that of the set risk measures and can be derived explicitly, important for actuarial users.
(3) MTCE is the natural extension of the classical TCE to the multivariate case. At the same time it takes into account the covariance structure of collective of risk sunder consideration.
The MTCE risk measure has many important properties, see details in subsection 3.2 .
Following the MTCE risk measure, we will introduce the multivariate tail covariance (MTCov) measure, which is a matrix-valued risk measure designed to explore the tail dispersion of multivariate loss distributions (Landsman.Z, Makov.U, Shushi.T - Insurance: Mathematics and Economics, 2018). The MTCov is the second multivariate tail conditional moment around the MTCE.
The multivariate tail covariance given $\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})$ is defined by

$$MTCov_{\mathbf{q}}(\mathbf{X}) = E((\mathbf{X} - MTCE_{\mathbf{q}}(\mathbf{X}))(\mathbf{X} - MTCE_{\mathbf{q}}(\mathbf{X}))^T|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})). \tag{1.3}$$

The MTCov risk measure has the following properties :

1. For any n$\times$ 1 random vector of risks $\mathbf{X}$ and positive constant λ, we

have

$$MTCov_{\mathbf{q}}(\lambda\mathbf{X}) = \lambda^2 MTCov_{\mathbf{q}}(\mathbf{X}).$$

2. For any $\mathbf{X}$ and any vector of constants $\boldsymbol{\alpha} \in R^n$

$$MTCov_q(\mathbf{X} + \boldsymbol{\alpha}) = MTCov_q(\mathbf{X}).$$

   This means that for a fixed amount of known loss α the dispersion of the total risk $\mathbf{X}+\boldsymbol{\alpha}$ is the same as the dispersion.

3. If the vector of risks $\mathbf{X}$ has independent components, then

$$MTCov_{\mathbf{q}}(\mathbf{X}) = diag(TV_{q_1}(X_1), TV_{q_2}(X_2), ..., TV_{q_n}(X_n)).$$

From the MTCov we can construct the multivariate tail correlation matrix, which defined as

$$MTCorr_{\mathbf{q}}(\mathbf{X})_{ik} = (\frac{MTCov_{\mathbf{q}}(\mathbf{X})_{ik}}{\sqrt{MTCov_{\mathbf{q}}(\mathbf{X})_{ii}}\sqrt{MTCov_{\mathbf{q}}(\mathbf{X})_{kk}}})_{i,k=1,...,n}.$$

Each element in the MTCorr matrix gives the value of the correlation coefficient between any two risks X_k and X_i, and is defined by

$$-1 \leq \rho_{ik} = \frac{MTCov_{\mathbf{q}}(\mathbf{X})_{ik}}{\sqrt{MTCov_{\mathbf{q}}(\mathbf{X})_{ii}}\sqrt{MTCov_{\mathbf{q}}(\mathbf{X})_{kk}}} \leq 1.$$

In addition, we found expressions for the different risk measures: MTCE, MTCov and MTCorr, for multivariate Pareto type II distribution .
Pareto type II distribution, is a heavy-tail probability distribution used in business, economics, actuarial science, queueing theory and internet traffic modeling (Landsman.Z, Makov.U, Shushi.T - Insurance: Mathematics and Economics, 2018).
Furthermore, multivariate Pareto type II distribution is defined by shape and scale parameters which are given by α and $\boldsymbol{\sigma}$=$(\sigma_1, \sigma_2, ..., \sigma_n)^T$, respectively. Let $\mathbf{X} = (X_1, ..., X_n)^T$ be an n-dimensional multivariate Pareto type II distribution, the survival function is given by:

$$\overline{F}_{\mathbf{X}}(\mathbf{x}) = (1 + \sum_{j=1}^{n} \frac{x_j}{\sigma_j})^{-\alpha},$$

where $\mathbf{x} = (x_1, ..., x_n)^T$. It is known that the marginal distribution of X_j, j=1,...,n follows an univariate type II Pareto distribution with parameters α

and σ_j.
The density function for multivariate Pareto type II distribution is given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = (-1)^n \frac{d^n \overline{F}_{\boldsymbol{X}}(\boldsymbol{x})}{dx_1 \dots dx_n}$$

$$= \frac{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)}{\sigma_1\sigma_2\dots\sigma_n}(1+\sum_{j=1}^{n}\frac{X_j}{\sigma_j})^{-\alpha-n}.$$

We got that the multivariate tail conditional expectation (MTCE) for multivariate Pareto type II distribution is given by

$$MTCE_{\boldsymbol{q}}(\boldsymbol{X}) = VaR_{\boldsymbol{q}}(\boldsymbol{X}) + \boldsymbol{\mu}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j}), \tag{1.4}$$

where,

$$\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_n)^T,$$
$$VaR_{\mathbf{q}}(\mathbf{X}) = (VaR_{q_1}(X_1), VaR_{q_2}(X_2), ..., VaR_{q_n}(X_n))^T.$$

In addition, the multivariate tail covariance matrix (MTCov) for multivariate Pareto type II distribution is given by

$$MTCov_{\mathbf{q}}(\mathbf{X}) = V * \begin{bmatrix} \frac{\sigma_1^2\alpha}{(\alpha-1)^2(\alpha-2)} & \cdots & \frac{\sigma_1\sigma_n}{(\alpha-1)^2(\alpha-2)} \\ \frac{\sigma_1\sigma_2}{(\alpha-1)^2(\alpha-2)} & \cdots & \frac{\sigma_2\sigma_n}{(\alpha-1)^2(\alpha-2)} \\ \vdots & \ddots & \vdots \\ \frac{\sigma_1\sigma_n}{(\alpha-1)^2(\alpha-2)} & \cdots & \frac{\sigma_n^2\alpha}{(\alpha-1)^2(\alpha-2)} \end{bmatrix}, \tag{1.5}$$

where,

$$V = (1+\sum_{j=1}^{n}\frac{VaR_{q_j}(x_j)}{\sigma_j})^2.$$

Furthermore, we obtained that the multivariate tail correlation matrix for multivariate Pareto type II distribution is defined as

$$\text{MTCorr} = \begin{bmatrix} 1 & \frac{1}{\alpha} & \cdots & \frac{1}{\alpha} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\alpha} & \frac{1}{\alpha} & \cdots & 1 \end{bmatrix}. \tag{1.6}$$

The multivariate Pareto type II has a crucial disadvantage: any univariate marginal distribution has the same shape parameter, which means that they have the same distribution up to a scaling parameter. However, it is difficult to believe that all risk components of some system have the same rate of decrease of the tail distribution for large risks. Moreover, the dependence structure of multivariate Pareto is quite poor, because it allows only equal correlations for each couple of risks and the independent univariate Pareto marginals do not belong to the multivariate family.
Chiragiev and Landsman suggested two new multivariate versions of Pareto distribution, whose univariate marginals are Pareto, but with different shape parameters. They also have a lucratively richer dependence structure, i.e., a flexible one.
The first model, which is called Multivariate Flexible Pareto type I (MFP(I)), is the distribution where the power parameters of marginals do not depend on the order of the components included in the model. For the second one, called Multivariate Flexible Pareto type II (MFP(II)), the power parameters are already dependent on the order of their marginals. Therefore, the first model might be considered more attractive; for the second, some important dependence attributes can be calculated in a simpler form. In our thesis we work on the second type of Multivariate Flexible Pareto distribution.
The survival function of Multivariate Flexible Pareto type II distribution is given by

$$\overline{F}_{\mathbf{X}}(\mathbf{x},\boldsymbol{\sigma},\boldsymbol{\nu}) = \prod_{i=1}^{n}(1+\sum_{j=i}^{n}\frac{x_j}{\sigma_j})^{-\gamma_i},$$

and we write $\mathbf{X} \sim MFP(II)_n(\boldsymbol{\sigma},\boldsymbol{\nu})$, where $\boldsymbol{\nu} = (\nu_1, ..., \nu_n)$, with

$$\nu_i = \sum_{j=1}^{i}\gamma_j, \qquad i = 1, 2, ..., n.$$

In our research study, we also tried to find the same risk measures: MTCE, MTCov, MTCorr, for multivariate flexible Pareto type II, in order to see if a different result would be obtained, that is, whether the correlation is different between any two risks.

This thesis is organized as follows, chapter 2 discusses the VaR, TCE and TV for univariate Pareto type II distribution. Chapter 3 deals with the MTCE, MTCov and MTCorr for multivariate Pareto type II distribution. Furthermore, this chapter deals the capital allocation for multivariate Pareto type II distribution.

After finding expressions for the different risk measures: TCE, TV, MTCE, MTCov and MTCorr, for Pareto distribution type II, we showed that the correlation between any two risks is the same and equal to $\frac{1}{\alpha}$, regardless of which particular two risks are involved. This unrealistic result drew us to express the same risk measures for another distribution called "Multivariate Flexible Pareto". Thus, in chapter 4 we discussed the TCE and MTCE for Multivariate Flexible Pareto type II distribution.

Chapter 2

Tail risk measures for univariate Pareto type II distribution

2.1 Pareto type II distribution

Pareto type II distribution, is a heavy-tail probability distribution used in business, economics, actuarial science, queuing theory and Internet traffic modeling. It is named after K. S. Lomax. Pareto type II distribution support begins at zero with shape and scale parameters α and $\sigma > 0$, respectively. The density function is given by:

$$f_Y(y) = \frac{\alpha}{\sigma}(1 + \frac{y}{\sigma})^{-(\alpha+1)}, y > 0.$$

The survival function is given by:

$$\overline{F}(y) = (1 + \frac{y}{\sigma})^{-\alpha}, y > 0.$$

The raw moments of interest are given by

$$\alpha_1(Y) = \frac{\sigma}{\alpha - 1}, \alpha > 1$$

$$\alpha_2(Y) = \frac{2\sigma^2}{(\alpha - 1)(\alpha - 2)}, \alpha > 2$$

or, generally, for k$\in Z^+$ and $\alpha > k$,

$$\alpha_k(Y) = \Gamma(k+1)\frac{\Gamma(\alpha - k)}{\Gamma(\alpha)}.$$

The variance is given by

$$\mu_2(Y) = \frac{\sigma^2\alpha}{(\alpha-1)^2(\alpha-2)}, \alpha > 2.$$

2.2 VaR and TCE for univarite pareto type II distribution

Consider a loss random variable X whose distributed Pareto type II,
$X \sim Pareto^{(1)}(\alpha, \sigma)$, with shape and scale parameters α and $\sigma > 0$, respectively, the density function is given by

$$f_X(x) = \frac{\alpha}{\sigma}(1 + \frac{x}{\sigma})^{-(\alpha+1)}, \; x > 0. \tag{2.1}$$

The survival function is given by

$$\overline{F}_X(x) = (1 + \frac{x}{\sigma})^{-\alpha}, \qquad x > 0. \tag{2.2}$$

The concept of value-at-risk (VaR) has become the standard risk measure used to evaluate risks. The VaR is the amount of capital is required to be ensured, with high degree of certainty, that the enterprise does not become technically insolvent.

The promotion of VaR has prompted the study of risk measures by numerous authors (see Wang (1996,1997), Wirch and Hardy (1999)). Over recent years the so-called *tail conditional expectation* (TCE) or TailVaR risk measure has become more and more popular among actuaries , because of several attractive properties. This risk measure is one of the main subjects we dealt them in this thesis.

The tail conditional expectation (TCE) is defined by

$$TCE_q(x) = E(X|X > x_q) = \frac{\int_{x_q}^{\infty} x f_X(x) dx}{\overline{F}_X(x_q)}, \tag{2.3}$$

and is interpreted as the expected worse losses. Given the loss will exceed a particular value x_q, generally referred to as the q-th quantile with

$$\overline{F}_X(x_q) = 1 - q,$$

the TCE which defined in eq.(2.3) gives the expected loss that can potentially be experienced. This index has been initially recommended by Artzner et al. (1999) to measure both market and nonmarket risks, presumably for a portfolio of investments. It gives a measure of a right-tail risk, one with which actuaries are very familiar because insurance contracts typically possess exposures subject to "low-frequency but large-losses" as pointed out by Wang (1998).

Theorem 2.2.1. *Let $X \sim Pareto^{(1)}(\alpha, \sigma)$ the tail conditional expectation (TCE) can be defined by*

$$TCE_q(x) = x_q(\frac{\alpha}{\alpha - 1}) + \frac{\sigma}{\alpha - 1}. \tag{2.4}$$

Proof. For Pareto distribution the survival function at x_q (x_q is the q-th quantile) is given by

$$\overline{F}_X(x_q) = (1 + \frac{x_q}{\sigma})^{-\alpha} = 1 - q.$$

We have to go back to the equation(2.3), and concentrate on calculating the meter by using integration by parts.

$$\int_{x_q}^{\infty} x f_X(x) dx = [-x\overline{F}_X(x)]_{x_q}^{\infty} - \int_{x_q}^{\infty} -\overline{F}_X(x) dx,$$

pay attention, by using Lopital's rule we get that

$$\lim_{x \to \infty} \overline{F}_X(x) = 0.$$

Therefore,

$$\begin{aligned}
[-x\overline{F}_X(x)]_{x_q}^{\infty} - \int_{x_q}^{\infty} -\overline{F}_X(x) dx &= x_q\overline{F}_X(x_q) + \int_{x_q}^{\infty} \overline{F}_X(x_q) dx \\
&= x_q\overline{F}_X(x_q) + \int_{x_q}^{\infty} (1 + \frac{x}{\sigma})^{-\alpha} dx \\
&= x_q\overline{F}_X(x_q) + [\frac{\sigma}{-\alpha + 1}(1 + \frac{x}{\sigma})^{-\alpha+1}]_{x_q}^{\infty} \\
&= x_q\overline{F}_X(x_q) - \frac{\sigma}{-\alpha + 1}(1 + \frac{x_q}{\sigma})^{-\alpha+1} \\
&= x_q(1 - q) - \frac{\sigma}{-\alpha + 1}(1 + \frac{x_q}{\sigma})^{-\alpha}(1 + \frac{x_q}{\sigma}).
\end{aligned}$$

Finally,

$$TCE_q(x) = \frac{x_q(1-q) + \frac{\sigma}{\alpha-1}(1-q)(1+\frac{x_q}{\sigma})}{1-q} = x_q + \frac{\sigma}{\alpha-1}(1+\frac{x_q}{\sigma}) = x_q(\frac{\alpha}{\alpha-1}) + \frac{\sigma}{\alpha-1}.$$

Furthermore,

$$\overline{F}_X(x_q) = 1-q = (1+\frac{x_q}{\sigma})^{-\alpha}$$
$$(1-q)^{-\frac{1}{\alpha}} = 1+\frac{x_q}{\sigma}.$$

Then,

$$x_q = \sigma((1-q)^{-\frac{1}{\alpha}} - 1). \tag{2.5}$$

□

2.3 Tail variance for univariate Pareto type II distribution

TCE is a conditional expectation and does not include information about deviation of the risk from its expectation in the upper tail. In order to overcome this problem, Furman and Landsman (2006) introduced two new risk measures, the (Conditional) tail variance and the (Conditional) tail variance Premium. Tail variance is a measure of variability on the right tail $X > x_q$, and it is merely the conditional variance of the risk X. The tail variance is defined by

$$TV_q(X) = Var(X|X > x_q) = E((X - TCE_q(X))^2|X > x_q). \tag{2.6}$$

In this section we expressed the tail variance (TV) of univariate Pareto distribution.

Theorem 2.3.1. *Let* $X \sim Pareto^{(1)}(\alpha, \sigma)$, *with shape and scale parameters* α *and* $\sigma > 0$, *respectively . The tail variance* $TV_q(X)$ *is given by:*

$$TV_q(X) = \frac{\alpha\sigma^2}{(\alpha-1)^2(\alpha-2)}(1-q)^{-\frac{2}{\alpha}} \tag{2.7}$$

$$= \frac{\alpha\sigma^2}{(\alpha-1)^2(\alpha-2)}(1+\frac{x_q}{\sigma})^2. \tag{2.8}$$

Proof.

$$\begin{aligned} TV_q(X) &= Var(X|X > x_q) = E((X - TCE_q(X))^2|X > x_q) \\ &= E(X^2|X > x_q) - E^2(X|X > x_q). \end{aligned}$$

By using Theorem 2.2.1 , we get

$$E^2(X|X > x_q) = (TCE_q(X))^2 = (x_q + \frac{\sigma}{\alpha-1}(1+\frac{x_q}{\sigma}))^2.$$

Thus, we only need to express $E(X^2|X > x_q)$

$$E(X^2|X > x_q) = \frac{\int_{x_q}^{\infty} x^2 f_X(x)dx}{\overline{F}_X(x_q)} = \frac{\int_{x_q}^{\infty} x^2 f_X(x)dx}{1-q}.$$

Denote by k the meter of the equation above, then

$$k = \int_{x_q}^{\infty} x^2 f_X(x)dx = [-x^2\overline{F}_X(x)]_{x_q}^{\infty} + 2\int_{x_q}^{\infty} x\overline{F}_X(x)dx = x_q^2\overline{F}_X(x_q) + 2\int_{x_q}^{\infty} x(1+\frac{x}{\sigma})^{-\alpha}dx$$

by using Lopital's rule, we get that

$$\lim_{x\to\infty} -x^2\overline{F}_X(x) = \lim_{x\to\infty} \frac{-x^2}{(1+\frac{x}{\sigma})^{\alpha}} = 0, \text{ in condition } \alpha > 2.$$

Denote by k' the integral in the second part of k

$$\begin{aligned} k' = \int_{x_q}^{\infty} x(1+\frac{x}{\sigma})^{-\alpha}dx &= [x\frac{\sigma}{-\alpha+1}(1+\frac{x}{\sigma})^{-\alpha+1}]_{x_q}^{\infty} - \frac{\sigma}{-\alpha+1}\int_{x_q}^{\infty}(1+\frac{x}{\sigma})^{-\alpha+1}dx \\ &= \frac{\sigma x_q}{\alpha-1}(1+\frac{x_q}{\sigma})^{-\alpha+1} - \frac{\sigma^2}{(-\alpha+1)(-\alpha+2)}[(1+\frac{x}{\sigma})^{-\alpha+2}]_{x_q}^{\infty} \\ &= \frac{\sigma x_q}{\alpha-1}(1+\frac{x_q}{\sigma})^{-\alpha+1} + \frac{\sigma^2}{(\alpha-1)(\alpha-2)}(1+\frac{x_q}{\sigma})^{-\alpha+2}. \end{aligned}$$

Thus,

$$k = \int_{x_q}^{\infty} x^2 f_X(x)dx = x_q^2(1-q) + \frac{2x_q\sigma}{\alpha-1}(1+\frac{x_q}{\sigma})^{-\alpha+1} + \frac{2\sigma^2}{(\alpha-1)(\alpha-2)}(1+\frac{x_q}{\sigma})^{-\alpha+2}.$$

Thus, it follows that

$$\begin{aligned} TV_q(X) &= E(X^2|X > x_q) - E^2(X|X > x_q) \\ &= \frac{k}{1-q} - (x_q + \frac{\sigma}{\alpha-1}(1+\frac{x_q}{\sigma}))^2 \\ &= \frac{\sigma^2}{\alpha-1}(1+\frac{x_q}{\sigma})^2(\frac{2}{\alpha-2} - \frac{1}{\alpha-1}) \\ &= \frac{\alpha\sigma^2}{(\alpha-1)^2(\alpha-2)}(1-q)^{\frac{-2}{\alpha}}. \end{aligned}$$

Where,

$$x_q = \sigma((1-q)^{\frac{-1}{\alpha}} - 1).$$

□

Chapter 3

Multivariate Tail risk measures of multivariate Pareto distribution

3.1 Multivariate Pareto type II distribution

The Pareto distribution was introduced by Pareto (1897) as a model for the distribution income. Arnold (1983) proposed four generalized multivariate Pareto distributions denoted $MP_{(n)}(I), ..., MP_{(n)}(IV)$, the first three being special cases of the fourth one. Recently, Yeh (2000, 2004) studied some properties and inference for all four forms, mentioning that the generalized multivariate Pareto distribution is expected to fit the upper tail of some multivariate continuous income data and socio-economic multivariate variables, while, in particular, the multivariate Pareto distribution of the second kind is suited in reliability for its truncation invariance property. Originated in extreme value theory, the Pareto distribution is also an important ingredient of many risk management problems related to insurance, reinsurance and finance, see e.g. Embrechts et al. (1999). Widely used in insurance to model univariate heavy-tailed claims, the Pareto distribution could also be an interesting alternative for multivariate losses. Therefore, a multivariate Pareto distribution may be used to analyze a whole system of Pareto distributed business lines, e.g. for evaluation of aggregate losses, for risk capital analysis and allocation, for portfolio optimization etc.

Multivariate Pareto type II distribution considered by Shape and scale parameters are given by α and $\boldsymbol{\sigma}=(\sigma_1, \sigma_2, ..., \sigma_n)^T$, respectively. Let $\mathbf{X} = (X_1, ..., X_n)^T$ be an n-dimensional multivariate Pareto type II distribution, the survival function is given by

$$\overline{F}_{\mathbf{X}}(\mathbf{x}) = (1 + \sum_{j=1}^{n} \frac{x_j}{\sigma_j})^{-\alpha},$$

where $\mathbf{x} = (x_1, ..., x_n)^T$. It is known that the marginal distribution of X_j, j=1,...,n follows an univariate type II Pareto distribution with parameters α and σ_j.

Furthermore, the dependence structure of the marginals is characterized by parameter α; that is the correlation between X_i and X_j, for $i \neq j$ is given by $\frac{1}{\alpha}$.

3.2 MTCE for Multivariate Pareto distribution

The MTCE (multivariate tail conditional expectation) risk measure takes into account the covariation between dependent risks, which is the case when we are dealing with real data of losses.
Where $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ is an $n \times 1$ vector of risks whose cumulative distribution function (cdf) and tail function are denoted by $F_{\mathbf{X}}(\mathbf{x})$ and $\overline{F}_{\mathbf{X}}(\mathbf{x})$, respectively. The MTCE is defined by

$$MTCE_{\mathbf{q}}(\mathbf{X}) = E(\mathbf{X}|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X}))$$

3.2.1 Main properties of the MTCE measure

The MTCE risk measure has the following properties:

1. **Positive Homogeneity:** For any n×1 random vector of risks $\mathbf{X}$ and positive constant λ, we have
$MTCE_q(\lambda\mathbf{X}) = \lambda MTCE_q(\mathbf{X})$.

2. **Translation Invariance:**For any n×1 random vector of risks $\mathbf{X}$ and any vector of constants $\boldsymbol{\alpha} \in R^n$
$MTCE_q(\mathbf{X} + \boldsymbol{\alpha}) = MTCE_q(\mathbf{X}) + \boldsymbol{\alpha}$.

3. **Independence of risks:**If the vector of risks $\mathbf{X}$ has independent components, then $MTCE_q(\mathbf{X}) = TCE_q(\mathbf{X})$.

4. **Monotonicity:** Suppose $\mathbf{Y}$,$\mathbf{X}$ are n × 1 random vectors of risks and $\mathbf{Y} \geq \mathbf{X}$.Then: $MTCE_q(\mathbf{Y} - \mathbf{X}) \geq \mathbf{0}$, where $\mathbf{0}$ is vector of n zeros.

5. **Semi-sub additivity for elliptical distributions:** Suppose $\mathbf{X}=(\mathbf{X}_1^T, \mathbf{X}_2^T)^T$ is 2n × 1 elliptically distributed vector , and $\mathbf{X}_1=(X_1, ..., X_n)^T$,
$\mathbf{X}_2 = (X_{n+1}, ..., X_{2n})^T$ are is partition.
Then, $MTCE_q(\mathbf{X}_1, \mathbf{X}_2) \leq MTCE_q(\mathbf{X})_1 + MTCE_q(\mathbf{X})_2$.
Here $MTCE_q(\mathbf{X})_1$ is the vector of the n elements of $MTCE_q(\mathbf{X})$ and $MTCE_q(\mathbf{X})_2$ is the vector of it's last n elements.The implication of this inequality is that the combined risks, given that $\mathbf{X}_1 > VaR_q(\mathbf{X}_1)$and

$\mathbf{X}_2 > VaR_q(\mathbf{X}_2)$, is less risky than treating the risk separately , implying a clear gain from diversification.In the case where $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent the semi-sub-additivity reduces to sub-additivity since $MTCE_q(\mathbf{X})_1 = MTCE_q(\mathbf{X}_1)$ and $MTCE_q(\mathbf{X})_2 = MTCE_q(\mathbf{X}_2)$.

3.3 MTCE of bivariate Pareto type II distribution

In this section we will express the multivariate tail conditional expectation of bivariate pareto type II distribution.

Lemma 3.3.1. *Let* $\boldsymbol{X} = (X_1, X_2)^T \sim MPareto^{(2)}(\alpha, \boldsymbol{\sigma})$ *with survival function denoted by* $\overline{F}_{\boldsymbol{X}}(\boldsymbol{x}; \alpha, \boldsymbol{\sigma})$ *,* $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ *, the multivariate tail conditional expectation can be expressed by:*

$$MTCE_{\boldsymbol{q}}(\boldsymbol{X}) = E(\boldsymbol{X}|\boldsymbol{X} > VaR_{\boldsymbol{q}}(\boldsymbol{X})) = VaR_{\boldsymbol{q}}(\boldsymbol{X}) + \boldsymbol{\mu}(1 + \sum_{i=1}^{2} \frac{VaR_{q_i}(x_i)}{\sigma_i}). \tag{3.1}$$

Here $VaR_{\boldsymbol{q}}(\boldsymbol{X})$ *is 2× 1 vector,* $VaR_{\boldsymbol{q}}(\boldsymbol{X}) = (VaR_{q_1}(X_1), VaR_{q_2}(X_2))^T$*, and* $VaR_{q_j}(X_j) = x_{q_j}$ *the value at risk of* X_j *under the* q_j*-th quantile,*
$q_j \in (0, 1)$*, j=1,2. Where* $\boldsymbol{q} = (q_1, q_2)^T$.

Proof.

$$\begin{aligned} MTCE_{\boldsymbol{q}}(\boldsymbol{X}) &= E(\boldsymbol{X}|\boldsymbol{X} > VaR_{\mathbf{q}}(\boldsymbol{X})) \\ &= E(\boldsymbol{X}|X_1 > VaR_{q_1}(X_1), X_2 > VaR_{q_2}(X_2)), \quad 0 < q_1, q_2 < 1 \\ &= \frac{\int_{\boldsymbol{x_q}}^{\infty} \boldsymbol{x} f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}}{\overline{F}_{\boldsymbol{X}}(\boldsymbol{x_q})}. \end{aligned}$$

The survival function is given by:

$$\overline{F}_{\boldsymbol{X}}(\boldsymbol{x}) = (1 + \sum_{i=1}^{2} \frac{x_i}{\sigma_i})^{-\alpha} = (1 + \frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2})^{-\alpha},$$

where,

$$\overline{F}_{\boldsymbol{X}}(VaR_{\boldsymbol{q}}(\boldsymbol{X})) = (1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}.$$

The density function is given by the following formula

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = (-1)^2 \frac{d^2 \overline{F}_{\boldsymbol{X}}(\boldsymbol{x})}{dx_1 dx_2}.$$

Firstly, we will present the development of the density function

$$\begin{aligned}
\overline{F}_{\boldsymbol{X}}(\boldsymbol{x}) &= (1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha}\\
\frac{d}{dx_1}\overline{F}_{\boldsymbol{X}}(\boldsymbol{x}) &= \frac{-\alpha}{\sigma_1}(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-1}\\
\frac{d^2}{dx_1dx_2}\overline{F}_{\boldsymbol{X}}(\boldsymbol{x}) &= \frac{-\alpha(-\alpha-1)}{\sigma_1\sigma_2}(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-2}\\
&= \frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-2}.
\end{aligned}$$

So, if $\boldsymbol{X} \sim MPareto^{(2)}(\alpha,\boldsymbol{\sigma})$ the density function is equal to

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-2}.$$

The $MTCE_{\mathbf{q}}(\mathbf{X})$ can be calculated by the following formula

$$MTCE_{\boldsymbol{q}}(\boldsymbol{X}) = \frac{\int_{\boldsymbol{x_q}}^{\infty}\boldsymbol{x}f_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}}{\overline{F}_{\boldsymbol{X}}(\boldsymbol{x_q})}. \tag{3.2}$$

We will focus on calculating the meter in eq(3.2)

$$\text{denote} \qquad I = \int_{\boldsymbol{x_q}}^{\infty} x_1 f_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}.$$

Then,

$$\begin{aligned}
I = \int_{\boldsymbol{x_q}}^{\infty} x_1 f_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x} &= \int_{VaR_{\boldsymbol{q}}(\boldsymbol{X})}^{\infty} x_1\frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-2}\,d\mathbf{x}\\
&= \int_{VaR_{q_2}(X_2)}^{\infty}\frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}(\int_{VaR_{q_1}(X_1)}^{\infty} x_1(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-2}dx_1)dx_2,
\end{aligned}$$

we will solve the internal integral above by using integration by parts method, thus

$$\begin{aligned}
I = \int_{VaR_{q_2}(X_2)}^{\infty}(\frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}&([x_1\frac{\sigma_1}{-\alpha-1}(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{(-\alpha-1)}]_{VaR_{q_1}(X_1)}^{\infty}\\
&-\frac{\sigma_1}{-\alpha-1}\int_{VaR_{q_1}(X_1)}^{\infty}(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-1}dx_1))dx_2.
\end{aligned}$$

Let us notice that:

$$\lim_{x_1\to\infty} x_1\frac{\sigma_1}{-\alpha-1}(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-1}=0.$$

Therefore,

$$I=\int_{VaR_{q_2}(X_2)}^{\infty}\frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}(VaR_{q_1}(X_1)\frac{\sigma_1}{\alpha+1}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-1}$$
$$+[\frac{\sigma_1^2}{-\alpha(\alpha+1)}(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha}]_{VaR_{q_1}(X_1)}^{\infty})dx_2$$

$$=\frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}\int_{VaR_{q_2}(X_2)}^{\infty}\frac{VaR_{q_1}(X_1)\sigma_1}{\alpha+1}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-1}$$
$$+\frac{\sigma_1^2}{\alpha(\alpha+1)}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha}dx_2$$

$$=\frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}([VaR_{q_1}(X_1)\frac{\sigma_1\sigma_2}{-\alpha(\alpha+1)}(1+\frac{VaR_{q_1(X_1)}}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha}]_{VaR_{q_2}(X_2)}^{\infty}$$
$$+\frac{\sigma_1^2\sigma_2}{\alpha(\alpha+1)(-\alpha+1)}([(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha+1}]_{VaR_{q_2}(X_2)}^{\infty})$$

$$=\frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}(VaR_{q_1}(X_1)\frac{\sigma_1\sigma_2}{\alpha(\alpha+1)}(1+\frac{VaR_{q_1(X_1)}}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}$$
$$+\frac{\sigma_1^2\sigma_2}{\alpha(\alpha+1)(\alpha-1)}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha+1})$$

$$=VaR_{q_1}(X_1)(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}+\frac{\sigma_1}{\alpha-1}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha+1},$$

note that,

$$\lim_{x_2\to\infty}\frac{\sigma_1\sigma_2VaR_{q_1}(X_1)}{-\alpha(\alpha+1)}(1+\frac{VaR_{q_1(X_1)}}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha}=0$$
$$\lim_{x_2\to\infty}\frac{\sigma_1^2\sigma_2}{\alpha(\alpha+1)(-\alpha+1)}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha+1}=0.$$

We conclude that:

$$\frac{\int_{\boldsymbol{x_q}}^{\infty} x_1 f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}}{\overline{F}_{\boldsymbol{X}}(\boldsymbol{x})}$$

$$= \frac{VaR_{q_1}(X_1)(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}+\frac{\sigma_1}{\alpha-1}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha+1}}{(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}}$$

$$= VaR_{q_1}(X_1) + \frac{\sigma_1}{\alpha-1}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2}).$$

Denote by II the following integral,

$$II = \int_{\boldsymbol{x_q}}^{\infty} x_2 f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x},$$

thus,

$$II = \int_{VaR_{\boldsymbol{q}}(\boldsymbol{X})}^{\infty} x_2 \frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-2}\, dx_2\, dx_1$$

$$= \int_{VaR_{q_1}(X_1)}^{\infty} \frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}(\int_{VaR_{q_2}(X_2)}^{\infty} x_2(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-2} dx_2) dx_1$$

$$= \frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}(\int_{VaR_{q_1}(X_1)}^{\infty}([x_2\frac{\sigma_2}{-\alpha-1}(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-1}]_{VaR_{q_2}(X_2)}^{\infty}$$
$$-\frac{\sigma_2}{-\alpha-1}\int_{VaR_{q_2(X_2)}}^{\infty}(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-1} dx_2) dx_1)$$

$$= \frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}\int_{VaR_{q_1}(X_1)}^{\infty} VaR_{q_2}(X_2)\frac{\sigma_2}{\alpha+1}(1+\frac{x_1}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha-1}$$
$$+\frac{\sigma_2^2}{-\alpha(\alpha+1)}[(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha}]_{VaR_{q_2}(X_2)}^{\infty} dx_1$$

$$= \frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}\int_{VaR_{q_1}(X_1)}^{\infty} VaR_{q_2}(X_2)\frac{\sigma_2}{\alpha+1}(1+\frac{x_1}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha-1}$$
$$+\frac{\sigma_2^2}{\alpha(\alpha+1)}(1+\frac{x_1}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}dx_1$$

$$= \frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}([\frac{\sigma_1\sigma_2}{-\alpha(\alpha+1)}VaR_{q_2}(X_2)(1+\frac{x_1}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}]_{VaR_{q_1}(X_1)}^{\infty}$$
$$+\frac{\sigma_2^2\sigma_1}{\alpha(\alpha+1)(-\alpha+1)}[(1+\frac{x_1}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha+1}]_{VaR_{q_1}(X_1)}^{\infty})$$

$$= \frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}(\frac{\sigma_1\sigma_2}{\alpha(\alpha+1)}VaR_{q_2}(X_2)(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}$$
$$+\frac{\sigma_2^2\sigma_1}{\alpha(\alpha+1)(\alpha-1)}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha+1})$$
$$= VaR_{q_2}(X_2)(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}+\frac{\sigma_2}{\alpha-1}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha+1}$$

So, we get

$$\frac{\int_{\boldsymbol{x_q}}^{\infty} x_2 f_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}}{\overline{F}_{\boldsymbol{X}}(\boldsymbol{x})}$$

$$= \frac{VaR_{q2}(X_2)(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}+\frac{\sigma_2}{\alpha-1}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha+1}}{(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}}$$
$$= VaR_{q2}(X_2)+\frac{\sigma_2}{\alpha-1}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2}).$$

Finally, if $\boldsymbol{X} \sim MPareto^{(2)}(\alpha, \boldsymbol{\sigma})$, the MTCE can be represented as

$$MTCE_{\boldsymbol{q}}(\boldsymbol{X}) = E(\boldsymbol{X}|\boldsymbol{X} > VaR_{\mathbf{q}}(\boldsymbol{X})) = VaR_{\boldsymbol{q}}(\boldsymbol{X}) + \boldsymbol{\mu}(1+\sum_{i=1}^{2}\frac{VaR_{q_i}(X_i)}{\sigma_i}),$$

where,

$$VaR_{\mathbf{q}}(\mathbf{X}) = (VaR_{q_1}(X_1), VaR_{q_2}(X_2))^T$$
$$VaR_{q_i}(X_i) = \frac{\sigma_i}{\alpha - 1}(\alpha((1 - q_i)^{-1/\alpha} - 1) + 1); i = 1, 2$$
$$\boldsymbol{\mu} = (\mu_1, \mu_2)^T = (\frac{\sigma_1}{\alpha - 1}, \frac{\sigma_2}{\alpha - 1})^T.$$

□

3.4 MTCE for multivariate Pareto type II distribution

In this section we will discuss a multivariatre Pareto type II distribution, and express the multivariate tail conditional expectation (MTCE) measure for it, under the assumption of different quantiles for the different risks.

Theorem 3.4.1. *Let $\boldsymbol{X} = (X_1, ..., X_n)^T$ is n×1 vector with a Pareto distribution, $X\sim$ $Pareto^{(n)}(\alpha, \boldsymbol{\sigma})$ where $\boldsymbol{\sigma}=(\sigma_1, \sigma_2, ..., \sigma_n)^T$, the multivariate tail conditional expectation(MTCE) can be defined as*

$$MTCE_{\boldsymbol{q}}(\boldsymbol{X}) = VaR_{\boldsymbol{q}}(\boldsymbol{X}) + \boldsymbol{\mu}(1 + \sum_{j=1}^{n} \frac{VaR_{q_j}(X_j)}{\sigma_j}), \tag{3.3}$$

where,

$$\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_n)^T,$$
$$VaR_{\boldsymbol{q}}(\boldsymbol{X}) = (VaR_{q_1}(X_1), VaR_{q_2}(X_2), ..., VaR_{q_n}(X_n))^T.$$

Proof. The density function for multivariate Pareto type II distribution is given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = (-1)^n \frac{d^n \overline{F}_{\boldsymbol{X}}(\boldsymbol{x})}{dx_1 \dots dx_n}$$

$$= \frac{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)}{\sigma_1\sigma_2\dots\sigma_n}(1 + \sum_{j=1}^{n} \frac{x_j}{\sigma_j})^{-\alpha-n}. \tag{3.4}$$

Where,

$$\begin{aligned} MTCE_{\boldsymbol{q}}(\boldsymbol{X}) &= E(\boldsymbol{X}|\boldsymbol{X} > VaR_{\mathbf{q}}(\boldsymbol{X})) \\ &= E(\boldsymbol{X}|X_1 > VaR_{q_1}(X_1), X_2 > VaR_{q_2}(X_2), \dots, X_n > VaR_{q_n}(X_n)) \\ &= \frac{\int_{VaR_{\mathbf{q}}(\mathbf{X})}^{\infty} \boldsymbol{x} f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}}{\overline{F}_{\boldsymbol{X}}(\boldsymbol{x_q})}, \quad 0 < q_1, \dots, q_n < 1 \end{aligned}$$

and the survival function is given by

$$\overline{F}_{\boldsymbol{X}}(\boldsymbol{x_q}) = (1 + \sum_{j=1}^{n} \frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha}.$$

We calculate n integrals of

$$I_i = \int_{VaR_{\mathbf{q}}(\mathbf{X})}^{\infty} x_i f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}.$$

Denote by c the density function constant

$$c = \frac{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)}{\sigma_1\sigma_2\dots\sigma_n}.$$

Thus,

$$I_i = \int_{VaR_{\mathbf{q}}(\mathbf{X})}^{\infty} x_i f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \quad = c\int_{VaR_{\mathbf{q}_{n-1,-i}}(\mathbf{X}_{n-1,-i})}^{\infty} \int_{VaR_{q_i}(X_i)}^{\infty} x_i (1+\sum_{i=1}^{n}\frac{x_i}{\sigma_i})^{-\alpha-n} d\mathbf{x}_n,$$

where,
$\mathbf{q}_{n-1,-i} = (q_1,\dots,q_{i-1},q_{i+1},\dots,q_n)^T$ and $\mathbf{X}_{n-1,-i} = (X_1,\dots,X_{i-1},X_{i+1},\dots,X_n)^T$.

By using integration by parts, we get that

$$I_i = c\int_{VaR_{\mathbf{q}_{n-1,-i}}(\mathbf{X}_{n-1,-i})}^{\infty} ([\frac{\sigma_i x_i}{-\alpha-n+1}(1+\sum_{j=1}^{n}\frac{x_j}{\sigma_j})^{-\alpha-n+1}]_{VaR_{q_i}(X_i)}^{\infty}$$
$$-\frac{\sigma_i}{-\alpha-n+1}\int_{VaR_{q_i}(X_i)}^{\infty}(1+\sum_{j=1}^{n}\frac{x_j}{\sigma_j})^{-\alpha-n+1}dx_i)d\mathbf{x}_{n-1,-i}$$

$$= c\int_{VaR_{\mathbf{q}_{n-1,-i}}(\mathbf{X}_{n-1,-i})}^{\infty} ([\frac{\sigma_i x_i}{-\alpha-n+1}(1+\sum_{j=1}^{n}\frac{x_j}{\sigma_j})^{-\alpha-n+1}]_{VaR_{q_i}(X_i)}^{\infty}$$
$$-[\frac{\sigma_i^2}{(-\alpha-n+1)(-\alpha-n+2)}(1+\sum_{j=1}^{n}\frac{x_j}{\sigma_j})^{-\alpha-n+2}]_{VaR_{q_i}(X_i)}^{\infty})d\mathbf{x}_{n-1,-i}$$

$$= c\int_{VaR_{\mathbf{q}_{n-1,-i}}(\mathbf{X}_{n-1,-i})}^{\infty} (\frac{\sigma_i}{\alpha+n-1}VaR_{q_i}(X_i)(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+1}$$
$$+\frac{\sigma_i^2}{(\alpha+n-1)(\alpha+n-2)}(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+2})d\mathbf{x}_{n-1,-i}$$

after doing (n-1) more integrals, we get that

$$I_i = c(\frac{\sigma_1\sigma_2\dots\sigma_n}{(\alpha+n-1)(\alpha+n-2)\dots(\alpha+n-n)}VaR_{q_i}(X_i)(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha-n+n}$$
$$+\frac{\sigma_i^2\sigma_1\dots\sigma_{i-1}\sigma_{i+1}\dots\sigma_n}{(\alpha+n-1)(\alpha+n-2)\dots(\alpha+n-(n+1))}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha-n+(n+1)})$$

$$= VaR_{q_i}(X_i)(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha} + \frac{\sigma_i}{\alpha-1}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha+1}.$$

Therefore, the i-th component of the vector $MTCE_{\mathbf{q}}(\mathbf{X})$ is given by

$$\frac{VaR_{q_i}(X_i)(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha} + \frac{\sigma_i}{\alpha-1}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha+1}}{(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha}}$$

$$= VaR_{q_i}(X_i) + \frac{\sigma_i}{\alpha-1}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j}).$$

So, for all j∈ 1,2,...,n the $(MTCE_{\mathbf{q}}(\mathbf{X}))_j$ is defined as

$$(MTCE_{\mathbf{q}}(\mathbf{X}))_j = VaR_{q_j}(X_j) + \mu_j(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j}), \qquad (3.5)$$

$$j = 1, ..., n$$

where $\mu_j = \frac{\sigma_j}{\alpha-1}$,

finally,

$$MTCE_{\mathbf{q}}(\mathbf{X}) = VaR_{\mathbf{q}}(\mathbf{X}) + \boldsymbol{\mu}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j}). \qquad (3.6)$$

□

3.5 Capital allocation for multivariate Pareto type II distribution

Let $\boldsymbol{X} = (X_1, ..., X_n)^T$ be the portfolio of Pareto risks, such that

$$X_i \sim Pareto(\alpha, \sigma_i), i = 1, ..., n.$$

Define by S a sum of risks

$$S = X_1 + X_2 + ... + X_n = \sum_{i=1}^{n} X_i.$$

The allocation for risk capital is defined as

$$\rho(S) = E(S|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})),$$

we get that

$$\begin{aligned}\rho(S) = E(S|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) &= E(\sum_{i=1}^{n} X_i|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) \\ &= \sum_{i=1}^{n} E(X_i|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})),\end{aligned}$$

where,

$$\rho(X_i) = E(X_i|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) = VaR_{q_i}(X_i) + \mu_i(1 + \sum_{i=1}^{n} \frac{VaR_{q_i}(X_i)}{\sigma_i}), i = 1, ..., n.$$

Thus, by using Theorem 3.4.1, we can see that

$$\rho(S) = \sum_{i=1}^{n} \rho(X_i) = \sum_{i=1}^{n} MTCE_{q_i}(X_i). \tag{3.7}$$

3.6 Multivariate tail covariance (MTCov) for multivariate Pareto type II distribution

In this section we will focusing on a multivariate tail covariance (MTCov) measure, which is a matrix-valued risk measure designed to explore the tail dispersion of multivariate loss distributions. The MTCov is the second multivariate tail conditional moment around the MTCE (multivariate tail conditional expectation). The multivariate tail covariance is considered by

$$MTCov_{\mathbf{q}}(\mathbf{X}) = E((\mathbf{X} - MTCE_{\mathbf{q}}(\mathbf{X}))(\mathbf{X} - MTCE_{\mathbf{q}}(\mathbf{X}))^T|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})). \tag{3.8}$$

We call this matrix the Multivariate Tail Covariance matrix, given $\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})$.

3.6.1 Main properties of the MTCov measure

The MTCov risk measure has the following properties :

1. For any n× 1 random vector of risks $\mathbf{X}$ and positive constant λ, we have

$$MTCov_{\mathbf{q}}(\lambda\mathbf{X}) = \lambda^2 MTCov_{\mathbf{q}}(\mathbf{X}).$$

2. For any $\mathbf{X}$ and any vector of constants $\boldsymbol{\alpha} \in R^n$

$$MTCov_q(\mathbf{X} + \boldsymbol{\alpha}) = MTCov_q(\mathbf{X}).$$

   This means that for a fixed amount of known loss α the dispersion of the total risk $\mathbf{X}+\boldsymbol{\alpha}$ is the same as the dispersion.

3. If the vector of risks $\mathbf{X}$ has independent components, then

$$MTCov_{\mathbf{q}}(\mathbf{X}) = diag(TV_{q_1}(X_1), TV_{q_2}(X_2), ..., TV_{q_n}(X_n)).$$

3.7 MTCov of bivariate Pareto type II distribution

Denote,

$$V = (1 + \sum_{i=1}^{2} \frac{VaR_{q_i}(X_i)}{\sigma_i})^2.$$

Lemma 3.7.1. *Let* $\boldsymbol{X} = (X_1, X_2)^T \sim MPareto^{(2)}(\alpha, \boldsymbol{\sigma})$ *with survival function denoted by* $\overline{F}_{\boldsymbol{X}}(\boldsymbol{x}; \alpha, \boldsymbol{\sigma})$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$, *the multivariate tail covariance matrix (MTCov) is given by*

$$MTCov_{\boldsymbol{q}}(\boldsymbol{X}) = V \times \begin{bmatrix} \frac{\sigma_1^2\alpha}{(\alpha-1)^2(\alpha-2)} & \frac{\sigma_1\sigma_2}{(\alpha-1)^2(\alpha-2)} \\ \frac{\sigma_1\sigma_2}{(\alpha-1)^2(\alpha-2)} & \frac{\sigma_2^2\alpha}{(\alpha-1)^2(\alpha-2)} \end{bmatrix}.$$

Proof. Denote a multivariate tail covariance matrix by A

$$A = MTCov_{\mathbf{q}}(\mathbf{X}) = E((\mathbf{X} - MTCE_{\mathbf{q}}(\mathbf{X}))(\mathbf{X} - MTCE_{\mathbf{q}}(\mathbf{X}))^T | \mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})). \tag{3.9}$$

Denote each of the diagonal elements in the multivariate tail covariance matrix by A_{ii}, we get

$$A_{ii} = Var(X_i | \mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) = E(X_i^2 | \mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) - E^2(X_i | \mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})), i = 1, 2. \tag{3.10}$$

The second part of the equation above is basically equal to $(MTCE_{\boldsymbol{q}}(\boldsymbol{X}))_i$ (see eq 3.5), so we will focus only on the first part of the last equation for i=1.

Denote

$$k' = E(X_1^2 | \mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})).$$

Then,

$$k' = E(X_1^2 | \mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) = \frac{\int_{VaR_{\mathbf{q}}(\mathbf{X})}^{\infty} x_1^2 f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}}{(1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}}. \tag{3.11}$$

Denote by c' the density function constant.

$$c' = \frac{\alpha(\alpha+1)}{\sigma_1\sigma_2},$$

and denote by k'' the numerator of k',

$$k'' = c' \int_{VaR_{q_2}(X_2)}^{\infty} \int_{VaR_{q_1}(X_1)}^{\infty} x_1^2 (1 + \frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2})^{-\alpha-2} dx_1 dx_2$$

$$k'' = c' \int_{VaR_{q_2}(X_2)}^{\infty} ([\frac{x_1^2 \sigma_1}{-\alpha-1}(1 + \frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2})^{-\alpha-1}]_{VaR_{q_1}(X_1)}^{\infty}$$
$$-\frac{2\sigma_1}{-\alpha-1} \int_{VaR_{q_1}(X_1)}^{\infty} x_1 (1 + \frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2})^{-\alpha-1} dx_1) dx_2$$

$$= c' \int_{VaR_{q_2}(X_2)}^{\infty} VaR_{q_1}^2(X_1) \frac{\sigma_1}{\alpha+1}(1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{x_2}{\sigma_2})^{-\alpha-1}$$
$$-\frac{2\sigma_1}{-\alpha-1}([\frac{\sigma_1 x_1}{-\alpha}(1 + \frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2})^{-\alpha}]_{VaR_{q_1}(X_1)}^{\infty} - [\frac{\sigma_1^2}{-\alpha(-\alpha+1)}(1 + \frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2})^{-\alpha+1}]_{VaR_{q_1}(X_1)}^{\infty}) dx_2$$

$$= c' \int_{VaR_{q_2}(X_2)}^{\infty} VaR_{q_1}^2(X_1) \frac{\sigma_1}{\alpha+1}(1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{x_2}{\sigma_2})^{-\alpha-1}$$
$$+2\sigma_1^2 \frac{VaR_{q_1}(X_1)}{\alpha(\alpha+1)}(1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{x_2}{\sigma_2})^{-\alpha}$$
$$+\frac{2\sigma_1^3}{\alpha(\alpha+1)(\alpha-1)}(1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{x_2}{\sigma_2})^{-\alpha+1} dx_2$$

$$= c'([\frac{\sigma_1\sigma_2}{-\alpha(\alpha+1)} VaR_{q_1}^2(X_1)(1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{x_2}{\sigma_2})^{-\alpha}]_{VaR_{q_2}(X_2)}^{\infty}$$
$$+[2VaR_{q_1}(X_1)\frac{\sigma_1^2\sigma_2}{\alpha(\alpha+1)(-\alpha+1)}(1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{x_2}{\sigma_2})^{-\alpha+1}]_{VaR_{q_2}(X_2)}^{\infty}$$
$$+[2\frac{\sigma_1^3\sigma_2}{\alpha(\alpha+1)(\alpha-1)(-\alpha+2)}(1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{x_2}{\sigma_2})^{-\alpha+2}]_{VaR_{q_2}(X_2)}^{\infty}).$$

Thus,

$$k' = \frac{k''}{(1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}}$$

$$= VaR_{q_1}^2(X_1) + VaR_{q_1}(x_1)\frac{2\sigma_1}{(\alpha-1)}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})$$
$$+\frac{2\sigma_1^2}{(\alpha-1)(\alpha-2)}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^2,$$

by using eq.(3.5), we get that

$$A_{11} = Var(X_1|\mathbf{X} > VaR_{\boldsymbol{q}}(\mathbf{X})) = E(X_1^2|\mathbf{X} > VaR_{\boldsymbol{q}}(\mathbf{X})) - E^2(X_1|\mathbf{X} > VaR_{\boldsymbol{q}}(\mathbf{X}))$$
$$= k' - (VaR_{q_1}(x_1) + \frac{\sigma_1}{\alpha-1}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2}))^2$$
$$= \frac{2\sigma_1^2}{(\alpha-1)(\alpha-2)}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^2 - \frac{\sigma_1^2}{(\alpha-1)^2}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^2$$
$$= \frac{\alpha\sigma_1^2}{(\alpha-1)^2(\alpha-2)}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^2.$$

In the same way, we get that

$$A_{22} = \frac{\alpha\sigma_2^2}{(\alpha-1)^2(\alpha-2)}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^2.$$

Now, we will express the off-diagonal elements of the MTCov matrix.

$$A_{12} = A_{21} = Cov(X_1, X_2|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X}))$$

$$= E(X_1X_2|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) - E(X_1|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X}))E(X_2|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})).$$

Let's focus on the first part of the last equation above. Denote by u' and u'' the following expressions

$$u' = \frac{\int_{VaR_{\mathbf{q}}(\mathbf{X})}^{\infty} x_1x_2 f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}}{(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}}$$
$$u'' = \int_{VaR_{\mathbf{q}}(\mathbf{X})}^{\infty} x_1x_2 f_{\mathbf{X}}(\mathbf{x})d\mathbf{x},$$

then,

$$
\begin{aligned}
u'' &= c'\int_{VaR_{q_2}(X_2)}^{\infty}(\int_{VaR_{q_1}(X_1)}^{\infty} x_1x_2(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-2}dx_1dx_2 \\
&= c'\int_{VaR_{q_2}(X_2)}^{\infty} x_2\int_{VaR_{q_1}(X_1)}^{\infty} x_1(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-2}dx_1)dx_2 \\
&= c'\int_{VaR_{q_2}(X_2)}^{\infty} x_2([\frac{x_1\sigma_1}{-\alpha-1}(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-1}]_{VaR_{q_1}(X_1)}^{\infty} \\
&\quad -[\frac{\sigma_1^2}{-\alpha(-\alpha-1)}(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha}]_{VaR_{q_1}(X_1)}^{\infty})dx_2 \\
&= c'\int_{VaR_{q_2}(X_2)}^{\infty} x_2VaR_{q_1}(X_1)\frac{\sigma_1}{\alpha+1}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha-1} \\
&\quad +\frac{x_2\sigma_1^2}{\alpha(\alpha+1)}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha}dx_2
\end{aligned}
$$

$$
\begin{aligned}
&= c'(VaR_{q_1}(X_1)\frac{\sigma_1}{\alpha+1}[\frac{x_2\sigma_2}{-\alpha}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha}]_{VaR_{q_2}(X_2)}^{\infty} \\
&\quad -\frac{\sigma_2^2}{-\alpha(-\alpha+1)}[(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha+1}]_{VaR_{q_2}(X_2)}^{\infty} \\
&\quad +\frac{\sigma_1^2}{\alpha(\alpha+1)}([\frac{x_2\sigma_2}{-\alpha+1}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha+1}]_{VaR_{q_2}(X_2)}^{\infty} \\
&\quad -\frac{\sigma_2^2}{(-\alpha+1)(-\alpha+2)}[(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{x_2}{\sigma_2})^{-\alpha+2}]_{VaR_{q_2}(X_2)}^{\infty}))
\end{aligned}
$$

$$
\begin{aligned}
&= c'(VaR_{q_1}(X_1)VaR_{q_2}(X_2)\frac{\sigma_1\sigma_2}{\alpha(\alpha+1)}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha} \\
&\quad +\frac{\sigma_1\sigma_2^2}{\alpha(\alpha-1)(\alpha+1)}VaR_{q_1}(X_1)(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha+1} \\
&\quad +\frac{\sigma_1^2\sigma_2}{\alpha(\alpha+1)(\alpha-1)}VaR_{q_2}(X_2)(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha+1} \\
&\quad +\frac{\sigma_1^2\sigma_2^2}{\alpha(\alpha+1)(\alpha-1)(\alpha-2)}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha+2}).
\end{aligned}
$$

So,

$$u' = \frac{u''}{(1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{VaR_{q_2}(X_2)}{\sigma_2})^{-\alpha}}$$

$$\begin{aligned}= VaR_{q_1}(X_1)VaR_{q_2}(X_2) + VaR_{q_1}(X_1)\frac{\sigma_2}{\alpha - 1}(1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{VaR_{q_2}(X_2)}{\sigma_2}) \\ +VaR_{q_2}(X_2)\frac{\sigma_1}{\alpha - 1}(1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{VaR_{q_2}(X_2)}{\sigma_2}) \\ +\frac{\sigma_1\sigma_2}{(\alpha - 1)(\alpha - 2)}(1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{VaR_{q_2}(X_2)}{\sigma_2})^2.\end{aligned}$$

We conclude that

$$A_{12} = \frac{\sigma_1\sigma_2}{(\alpha - 1)^2(\alpha - 2)}(1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{VaR_{q_2}(X_2)}{\sigma_2})^2.$$

Finally,

$$A = MTCov_{\mathbf{q}}(\mathbf{X}) = V \times \begin{bmatrix} \frac{\sigma_1^2\alpha}{(\alpha-1)^2(\alpha-2)} & \frac{\sigma_1\sigma_2}{(\alpha-1)^2(\alpha-2)} \\ \frac{\sigma_1\sigma_2}{(\alpha-1)^2(\alpha-2)} & \frac{\sigma_2^2\alpha}{(\alpha-1)^2(\alpha-2)} \end{bmatrix}.$$

□

3.8 MTCov for multivariate Pareto type II distribution

In this section we will show a multivariate tail covariance matrix for multivariate pareto type II distribution.

Denote,

$$V = (1 + \sum_{j=1}^{n} \frac{VaR_{q_j}(x_j)}{\sigma_j})^2.$$

Theorem 3.8.1. *Let* $\boldsymbol{X} = (X_1, ..., X_n)^T$ *be a vector with a Pareto distribution,* $\boldsymbol{X} \sim Pareto^{(n)}(\alpha, \boldsymbol{\sigma})$ *where* $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_n)^T$*, the multivariate tail covariance matrix given* $\boldsymbol{X} > VaR_{\boldsymbol{q}}(\boldsymbol{X})$ *is given by:*

$$MTCov_{\boldsymbol{q}}(\boldsymbol{X}) = V * \begin{bmatrix} \frac{\sigma_1^2\alpha}{(\alpha-1)^2(\alpha-2)} & \cdots & \frac{\sigma_1\sigma_n}{(\alpha-1)^2(\alpha-2)} \\ \frac{\sigma_1\sigma_2}{(\alpha-1)^2(\alpha-2)} & \cdots & \frac{\sigma_2\sigma_n}{(\alpha-1)^2(\alpha-2)} \\ \vdots & \ddots & \vdots \\ \frac{\sigma_1\sigma_n}{(\alpha-1)^2(\alpha-2)} & \cdots & \frac{\sigma_n^2\alpha}{(\alpha-1)^2(\alpha-2)} \end{bmatrix}. \tag{3.12}$$

Denote the multivariate tail covariance matrix by A,

$$A = MTCov_{\boldsymbol{q}}(\boldsymbol{X}),$$

here,

$$A_{ii} = \frac{\sigma_i^2\alpha}{(\alpha-1)^2(\alpha-2)}(1 + \sum_{j=1}^{n} \frac{VaR_{q_j}(X_j)}{\sigma_j})^2, \quad for \ .i \in 1, ..., n$$

$$A_{ik} = \frac{\sigma_i\sigma_k}{(\alpha-1)^2(\alpha-2)}(1 + \sum_{j=1}^{n} \frac{VaR_{q_j}(X_j)}{\sigma_j})^2,$$
$$i \neq k \quad for \quad i, k \in 1, ..., n.$$

Proof.

$$A_{ii} = Var(X_i|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X}))$$

$$= E(X_i^2|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) - E^2(X_i|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})). \qquad (3.13)$$

The second part of the eq.(3.13) can be taken from the MTCE vector (see 3.5). Then we will focus on the first part of the last equation.

Denote,

$$k* = E(X_i^2|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X}))$$
$$K** = \int_{VaR_{\mathbf{q}_{n-1,-i}}(\mathbf{X}_{n-1,-i})}^{\infty} \int_{VaR_{q_i}(X_i)}^{\infty} x_i^2 f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}.$$

We will focus on calculating $k**$

$$K** = c\int_{VaR_{\mathbf{q}_{n-1,-i}}(\mathbf{X}_{n-1,-i})}^{\infty} ([\frac{x_i^2\sigma_i}{-\alpha-n+1}(1+\sum_{j=1}^{n}\frac{x_j}{\sigma_j})^{-\alpha-n+1}]_{VaR_{q_i}(X_i)}^{\infty}$$
$$-\frac{2\sigma_i}{-\alpha-n+1}\int_{VaR_{q_i}(X_i)}^{\infty} x_i(1+\sum_{j=1}^{n}\frac{x_j}{\sigma_j})^{-\alpha-n+1}dx_i)d\mathbf{x}_{n-1,-i}$$

$$= c\int_{VaR_{\mathbf{q}_{n-1,-i}}(\mathbf{X}_{n-1,-i})}^{\infty} (\frac{\sigma_i}{\alpha+n-1}VaR_{q_i}^2(X_i)(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+1}$$
$$-\frac{2\sigma_i}{-\alpha-n+1}([\frac{x_i\sigma_i}{-\alpha-n+2}(1+\sum_{j=1}^{n}\frac{x_j}{\sigma_j})^{-\alpha-n+2}]_{VaR_{q_i}(X_i)}^{\infty}$$
$$-\frac{\sigma_i}{-\alpha-n+2}\int_{VaR_{q_i}(X_i)}^{\infty}(1+\sum_{j=1}^{n}\frac{x_j}{\sigma_j})^{-\alpha-n+2}))d\mathbf{x}_{n-1,-i}$$

$$= c\int_{VaR_{\mathbf{q}_{n-1,-i}}(\mathbf{X}_{n-1,-i})}^{\infty} (\frac{\sigma_i}{\alpha+n-1}VaR_{q_i}^2(X_i)(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+1}$$
$$+2VaR_{q_i}(X_i)\frac{\sigma_i^2}{(\alpha+n-1)(\alpha+n-2)}(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+2}$$
$$+2\frac{\sigma_i^3}{(\alpha+n-1)(\alpha+n-2)(\alpha+n-3)}(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+3})d\mathbf{x}_{n-1,-i}$$

$$= c\int_{VaR_{\mathbf{q}_{n-2,-(i,k)}}(\mathbf{X}_{n-2,-(i,k)})}^{\infty}([\frac{VaR^2_{q_i}(X_i)\sigma_i\sigma_k}{(\alpha+n-1)(-\alpha-n+2)}(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+2}$$

$$+\frac{2\sigma_i}{\alpha+n-1}(\frac{VaR_{q_i}(X_i)\sigma_i\sigma_k}{(\alpha+n-2)(-\alpha-n+3)}(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+3}$$

$$+\frac{\sigma_i^2\sigma_j}{(\alpha+n-2)(\alpha+n-3)(-\alpha-n+4)}(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+4})]^{\infty}_{VaR_{q_k}(X_k)})d\mathbf{x}_{n-2,-(i,k)}$$

$$= c\int_{VaR_{\mathbf{q}_{n-2,-(i,k)}}(\mathbf{X}_{n-2,-(i,k)})}^{\infty}(VaR^2_{q_i}(X_i)(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\frac{VaR_{q_k}(X_k)}{\sigma_k}+\sum_{j\neq i,k}\frac{x_j}{\sigma_j})^{-\alpha-n+2}$$

$$+\frac{2\sigma_i}{(\alpha+n-3)}VaR_{q_i}(X_i)(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\frac{VaR_{q_k}(X_k)}{\sigma_k}+\sum_{j\neq i,k}\frac{x_j}{\sigma_j})^{-\alpha-n+3}$$

$$+\frac{2\sigma_i^2}{(\alpha+n-3)(\alpha+n-4)}(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\frac{VaR_{q_k}(X_k)}{\sigma_k}+\sum_{j\neq i,k}\frac{x_j}{\sigma_j})^{-\alpha-n+4})d\mathbf{x}_{n-2,-(i,k)}.$$

In the n-th integral we will get

$$k** = c(VaR^2_{q_i}(X_i)\frac{\sigma_1\sigma_2\ldots\sigma_n}{(\alpha+n-1)(\alpha+n-2)\ldots\alpha}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha}$$

$$+2\frac{\sigma_i^2\sigma_2\ldots\sigma_n}{(\alpha+n-1)(\alpha+n-2)\ldots\alpha(\alpha-1)}VaR_{q_i}(X_i)(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha+1}$$

$$+2\frac{\sigma_i^3\sigma_2\ldots\sigma_n}{(\alpha+n-1)(\alpha+n-2)\ldots\alpha(\alpha-1)(\alpha-2)}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha+2})$$

$$= VaR^2_{q_i}(X_i)(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha}+\frac{2\sigma_i}{\alpha-1}VaR_{q_i}(X_i)(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha+1}$$

$$+\frac{2\sigma_i^2}{(\alpha-1)(\alpha-2)}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha+2}.$$

So,

$$k* = VaR^2_{q_i}(X_i)+\frac{2\sigma_i}{\alpha-1}VaR_{q_i}(X_i)(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})+\frac{2\sigma_i^2}{(\alpha-1)(\alpha-2)}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^2. \quad (3.14)$$

From $eq.(3.5)$ and $eq.(3.14)$ follows that

$$A_{ii} = E(X_i^2|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) - E^2(X_i|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X}))$$
$$= k * -((MTCE_{\mathbf{q}}(\mathbf{X}))_i)^2$$
$$= \frac{\sigma_i^2\alpha}{(\alpha-1)^2(\alpha-2)}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^2.$$

Finally, for all i$\in 1,2,\ldots,n$, the following equation exists

$$A_{ii} = E(X_i^2|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) - E^2(X_i|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X}))$$

$$= \frac{\sigma_i^2\alpha}{(\alpha-1)^2(\alpha-2)}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^2, \qquad (3.15)$$

$$i \in 1,2,\ldots,n.$$

Now, we will express A_{ik}

$$A_{ik} = Cov(X_i, X_k|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X}))$$

$$= E(X_iX_k|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) - E(X_i|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X}))E(X_k|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})). \qquad (3.16)$$

The second part of the $eq.(3.16)$ can be taken from the MTCE vector ($Theorem 3.4.1$), then we will focus on the first part of the last equation.

Denote,

$$u* = E(X_iX_k|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})).$$
$$u** = \int_{VaR_{q_n}(X_n)}^{\infty} \cdots \int_{VaR_{q_1}(X_1)}^{\infty} x_ix_kf_{\mathbf{X}}(\mathbf{x})dx_1\ldots dx_n.$$

So,

$$u** = c\int_{VaR_{q_n}(X_n)}^{\infty} \cdots \int_{VaR_{q_1}(X_1)}^{\infty} x_ix_k(1+\sum_{j=1}^{n}\frac{x_j}{\sigma_j})^{-\alpha-n}dx_1\ldots dx_n$$

$$= c\int_{VaR_{\mathbf{q}_{n-1,-i}}(\mathbf{X}_{n-1,-i})}^{\infty}(x_k[\frac{x_i\sigma_i}{-\alpha-n+1}(1+\sum_{j=1}^{n}\frac{x_j}{\sigma_j})^{-\alpha-n+1}]_{VaR_{q_i}(X_i)}^{\infty}$$
$$-x_k\frac{\sigma_i}{-\alpha-n+1}\int_{VaR_{q_i}(X_i)}^{\infty}(1+\sum_{j=1}^{n}\frac{x_j}{\sigma_j})^{-\alpha-n+1}dx_i)d\mathbf{x}_{n-1,-i}$$

$$= c\int_{VaR_{\mathbf{q}_{n-1,-i}}(\mathbf{X}_{n-1,-i})}^{\infty}(x_k VaR_{q_i}(X_i)\frac{\sigma_i}{\alpha+n-1}(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+1}$$
$$+\frac{\sigma_i^2 x_k}{(\alpha+n-1)(\alpha+n-2)}(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+2})d\mathbf{x}_{n-1,-i}$$

$$= c\int_{VaR_{\mathbf{q}_{n-2,-(i,k)}}(\mathbf{X}_{n-2,-(i,k)})}^{\infty}(\frac{\sigma_i VaR_{q_i}(X_i)}{(\alpha+n-1)}[\frac{\sigma_k x_k}{-\alpha-n+2}(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+2}]_{VaR_{q_k}(X_k)}^{\infty}$$
$$+\frac{\sigma_i\sigma_k^2 VaR_{q_i}(X_i)}{(\alpha+n-1)(-\alpha-n+2)(-\alpha-n+3)}(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+3}$$
$$+\frac{\sigma_i^2}{(\alpha+n-1)(\alpha+n-2)}[\frac{\sigma_k x_k}{-\alpha-n+3}(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+3}]_{VaR_{q_k}(X_k)}^{\infty}$$
$$+\frac{\sigma_i^2\sigma_k^2}{(\alpha+n-1)(\alpha+n-2)(-\alpha-n+3)(-\alpha-n+4)}(1+\frac{VaR_{q_i}(X_i)}{\sigma_i}+\sum_{j\neq i}\frac{x_j}{\sigma_j})^{-\alpha-n+4})d\mathbf{x}_{n-2,-(i,k)}$$

$$c\int_{VaR_{\mathbf{q}_{n-2,-(i,k)}}(\mathbf{X}_{n-2,-(i,k)})}^{\infty}(\frac{\sigma_i\sigma_k VaR_{q_i}(X_i)VaR_{q_k}(X_k)}{(\alpha+n-1)(\alpha+n-2)}(1+\sum_{j=i,k}\frac{x_j}{\sigma_j}+\sum_{j\neq i,k}\frac{x_j}{\sigma_j})^{-\alpha-n+2}$$
$$+\frac{\sigma_i\sigma_k^2 VaR_{q_i}(X_i)}{(\alpha+n-1)(\alpha+n-2)(\alpha+n-3)}(1+\sum_{j=i,k}\frac{x_j}{\sigma_j}+\sum_{j\neq i,k}\frac{x_j}{\sigma_j})^{-\alpha-n+3}$$
$$+\frac{\sigma_i^2\sigma_k VaR_{q_k}(X_k)}{(\alpha+n-1)(\alpha+n-2)(\alpha+n-3)}(1+\sum_{j=i,k}\frac{x_j}{\sigma_j}+\sum_{j\neq i,k}\frac{x_j}{\sigma_j})^{-\alpha-n+3}$$
$$+\frac{\sigma_i^2\sigma_k^2}{(\alpha+n-1)(\alpha+n-2)(\alpha+n-3)(\alpha+n-4)}(1+\sum_{j=i,k}\frac{x_j}{\sigma_j}+\sum_{j\neq i,k}\frac{x_j}{\sigma_j})^{-\alpha-n+4})d\mathbf{x}_{n-2,-(i,k)}.$$

In the n-th integral we will get

$$u** = c(VaR_{q_i}(X_i)VaR_{q_k}(X_k)\frac{\sigma_1\sigma_2\ldots\sigma_n}{(\alpha+n-1)(\alpha+n-2)\ldots\alpha}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha}$$

$$+VaR_{q_i}(X_i)\frac{\sigma_i\sigma_k^2\prod_{j\neq i,k}\sigma_j}{(\alpha+n-1)(\alpha+n-2)\ldots\alpha(\alpha-1)}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha+1}$$

$$+VaR_{q_k}(X_k)\frac{\sigma_i^2\sigma_k\prod_{j\neq i,k}\sigma_j}{(\alpha+n-1)(\alpha+n-2)\ldots\alpha(\alpha-1)}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha+1}$$

$$+\frac{\sigma_i^2\sigma_k^2\prod_{j\neq i,k}\sigma_j}{(\alpha+n-1)(\alpha+n-2)\ldots\alpha(\alpha-1)(\alpha-2)}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^{-\alpha+2}).$$

Finally,

$$u* = VaR_{q_i}(X_i)VaR_{q_k}(X_k)+VaR_{q_i}(X_i)\frac{\sigma_k}{\alpha-1}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})$$

$$+VaR_{q_k}(X_k)\frac{\sigma_i}{\alpha-1}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})+\frac{\sigma_i\sigma_k}{(\alpha-1)(\alpha-2)}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^2.$$

We obtain from $eq.$(3.17) and $eq.$(3.6) the following expression for A_{ik}

$$A_{ik} = u*-(MTCE_{\mathbf{q}}(\mathbf{X}))_i(MTCE_{\mathbf{q}}(\mathbf{X}))_k$$
$$= \frac{\sigma_i\sigma_k}{(\alpha-1)^2(\alpha-2)}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^2.$$

Finally, for all i, k $\in 1,2,\ldots,n$, the following equation exists

$$A_{ik} = E(X_iX_k|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) - E(X_i|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X}))E(X_k|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X}))$$

$$= \frac{\sigma_i\sigma_k}{(\alpha-1)^2(\alpha-2)}(1+\sum_{j=1}^{n}\frac{VaR_{q_j}(X_j)}{\sigma_j})^2, \qquad (3.17)$$

where $\quad i\neq k; \quad i,k\in 1,2,\ldots,n.$

□

3.9 MTCorr for multivariate Pareto distribution

The correlation between each two variables must be expressed by using the most familiar measure of dependence between two quantities is the Pearson product-moment correlation coefficient, or "Pearson's correlation coefficient". It is obtained by dividing the covariance of the two variables by the product of their standard deviations.

$$MTCorr_{\mathbf{q}}(\mathbf{X})_{ik} = \left(\frac{MTCov_{\mathbf{q}}(\mathbf{X})_{ik}}{\sqrt{MTCov_{\mathbf{q}}(\mathbf{X})_{ii}}\sqrt{MTCov_{\mathbf{q}}(\mathbf{X})_{kk}}}\right)_{i,k=1,\ldots,n}.$$

From well-known Cauchy-Schwartz inequality it may be easily shown that for elements of this matrix the following inequality holds

$$-1 \leq \rho_{ik} = \frac{MTCov_{\mathbf{q}}(\mathbf{X})_{ik}}{\sqrt{MTCov_{\mathbf{q}}(\mathbf{X})_{ii}}\sqrt{MTCov_{\mathbf{q}}(\mathbf{X})_{kk}}} \leq 1.$$

Thus, in this section we will present the tail correlation matrix for multivariate Pareto distribution.

Theorem 3.9.1. *Let* $\boldsymbol{X} = (X_1, ..., X_n)^T$ *be a vector with a Pareto distribution,* $\boldsymbol{X} \sim Pareto^{(n)}(\alpha, \boldsymbol{\sigma})$*, where* $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)^T$*, the multivariate tail correlation matrix, given* $\boldsymbol{X} > VaR_{\boldsymbol{q}}(\boldsymbol{X})$ *is given by*

$$MTCorr = \begin{bmatrix} 1 & \frac{1}{\alpha} & \cdots & \frac{1}{\alpha} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\alpha} & \frac{1}{\alpha} & \cdots & 1 \end{bmatrix}. \tag{3.18}$$

Proof. By using Theorem (3.8.1), we will get

$$\rho_{X_i,X_k} = \frac{A_{ik}}{\sqrt{A_{ii}}\sqrt{A_{kk}}} = \frac{1}{\alpha},$$

$$for \ \ i \neq k \ \ i,k = 1,2,...,n.$$

Finally,

$$\text{MTCorr} = \begin{bmatrix} 1 & \frac{1}{\alpha} & \cdots & \frac{1}{\alpha} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\alpha} & \frac{1}{\alpha} & \cdots & 1 \end{bmatrix}.$$

□

3.10 Numerical illustration

The suggested formulas above can easily be evaluated in any computer program environment. We included 3 examples of MTCE, MTCov and MTCorr for 5 business lines having multivariate dependent Pareto distribution type II, with n=10000 observations everyone, performed in R Studio environment.

Example 1. *Let* $\boldsymbol{X} = (X_1, X_2, ..., X_5)^T \sim Pareto^{(5)}(\alpha, \boldsymbol{\sigma})$*, with the following parameters:* $\alpha = 2.1$*,* $\boldsymbol{\sigma} = (2.1, 2.5, 2.8, 3.5, 5)^T$ *and* $\boldsymbol{q} = (0.95, 0.95, 0.97, 0.96, 0.94)^T$*. By using eq.(3.6), we obtained*

$$MTCE_{\boldsymbol{q}}(\boldsymbol{X}) = (32.84719, 35.88925, 39.59785, 45.81910, 61.51359)^T,$$

also by using eq.(3.13) and (3.16), we obtained

$$MTCov = \begin{bmatrix} 12381.845 & 7019.187 & 7861.489 & 9826.861 & 14038.37 \\ 7019.187 & 17547.966 & 9358.915 & 11698.644 & 16712.35 \\ 7861.489 & 9358.915 & 22012.169 & 13102.482 & 18717.83 \\ 9826.861 & 11698.644 & 13102.482 & 34394.014 & 23397.29 \\ 14038.373 & 16712.349 & 18717.831 & 23397.289 & 70191.87 \end{bmatrix},$$

and by using eq.(3.18), we got

$$MTCorr = \begin{bmatrix} 1 & 0.4761905 & 0.4761905 & 0.4761905 & 0.4761905 \\ 0.4761905 & 1 & 0.4761905 & 0.4761905 & 0.4761905 \\ 0.4761905 & 0.4761905 & 1 & 0.4761905 & 0.4761905 \\ 0.4761905 & 0.4761905 & 0.4761905 & 1 & 0.4761905 \\ 0.4761905 & 0.4761905 & 0.4761905 & 0.4761905 & 1 \end{bmatrix}$$

also by using eq.(3.7). we obtained that $\rho(s) = 215.667$*, we can see there is a positive relationship between the risk's contribution to the capital allocation and the scale parameter.*

Example 2. *Let* $\boldsymbol{X} = (X_1, X_2, ..., X_5)^T \sim Pareto^{(5)}(\alpha = 3, \boldsymbol{\sigma})$*,* $\boldsymbol{\sigma}$ *is given in the previous example, and by using the same equations, we got,*

$$MTCE = (30.86475, 32.15155, 35.15528, 38.68951, 49.63893)^T,$$

and the MTCov is equal to

$$MTCov = \begin{bmatrix} 1041.0800 & 413.1270 & 462.7022 & 578.3778 & 826.2540 \\ 413.1270 & 1475.4535 & 550.8360 & 688.5450 & 983.6357 \\ 462.7022 & 550.8360 & 1850.8089 & 771.1704 & 1101.6720 \\ 578.3778 & 688.5450 & 771.1704 & 2891.8889 & 1377.0900 \\ 826.2540 & 983.6357 & 1101.6720 & 1377.0900 & 5901.8142 \end{bmatrix},$$

also we obtained the MTCorr matrix which given by

$$MTCorr = \begin{bmatrix} 1 & 0.333 & 0.333 & 0.333 & 0.333 \\ 0.333 & 1 & 0.333 & 0.333 & 0.333 \\ 0.333 & 0.333 & 1 & 0.333 & 0.333 \\ 0.333 & 0.333 & 0.333 & 1 & 0.333 \\ 0.333 & 0.333 & 0.333 & 0.333 & 1 \end{bmatrix},$$

and by using eq.(3.7) we obtained that $\rho(s)$=186.5. In comparison with the first example, using the same scale parameters with $\alpha = 3$, we got that all the tail measures received more lower values than the values they were received before.

Example 3. *Let $\boldsymbol{X} = (X_1, X_2, ..., X_5)^T \sim Pareto^{(5)}(\alpha, \boldsymbol{\sigma})$, where, $\alpha = 3$, $\boldsymbol{\sigma} = (2.5, 2.8, 3, 4, 5.8)^T$, by using the same equations above, we got,*

$$MTCE = (26.95933, 28.02023, 30.12699, 34.29942, 44.76377)^T,$$

and the MTCov is equal to

$$MTCov = \begin{bmatrix} 886.2685 & 330.8736 & 354.5074 & 472.6765 & 685.3809 \\ 330.8736 & 1111.7352 & 397.0483 & 529.3977 & 767.6267 \\ 354.5074 & 397.0483 & 1276.2266 & 567.2118 & 822.4571 \\ 472.6765 & 529.3977 & 567.2118 & 2268.8473 & 1096.6095 \\ 685.3809 & 767.6267 & 822.4571 & 1096.6095 & 4770.2514 \end{bmatrix}.$$

The MTCorr is given by

$$MTCorr = \begin{bmatrix} 1 & 0.333 & 0.333 & 0.333 & 0.333 \\ 0.333 & 1 & 0.333 & 0.333 & 0.333 \\ 0.333 & 0.333 & 1 & 0.333 & 0.333 \\ 0.333 & 0.333 & 0.333 & 1 & 0.333 \\ 0.333 & 0.333 & 0.333 & 0.333 & 1 \end{bmatrix},$$

and by using eq.(3.7), we got that the allocation is equal to 186.5.

Then, after increasing the values of all the scale parameters, with $\alpha = 3$, all the tail measures received more lower values than the values they received in the previous example, except, the MTCorr which only depends on the shape parameter value, then, we got the same MTCorr matrix in the last two examples.

Example 4. *Let* $\boldsymbol{X} = (X_1, X_2, ..., X_5)^T \sim Pareto^{(5)}(\alpha, \boldsymbol{\sigma})$*, where,* $\alpha = 4$*,* $\boldsymbol{\sigma} = (2.3, 2.1, 4, 8, 7)^T$*, by using the same equations above, we got,*

$$MTCE = (29.17432, 30.17465, 35.12573, 57.28006, 50.84514)^T,$$

and the MTCov is equal to

$$MTCov = \begin{bmatrix} 434.34594 & 99.14418 & 188.8461 & 377.6921 & 330.4806 \\ 99.14418 & 362.09180 & 172.4247 & 344.8493 & 301.7432 \\ 188.84606 & 172.42467 & 1313.7117 & 656.8559 & 574.7489 \\ 377.69212 & 344.84933 & 656.8559 & 5254.8470 & 1149.4978 \\ 330.48061 & 301.74316 & 574.7489 & 1149.4978 & 4023.2422 \end{bmatrix}.$$

The MTCorr is given by

$$MTCorr = \begin{bmatrix} 1 & 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 1 & 0.25 & 0.25 & 0.25 \\ 0.25 & 00.25 & 1 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 1 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 & 1 \end{bmatrix},$$

and by using eq.(3.7), we got that the allocation is equal to 202.6 .

The figures on the next pages describe the allocation for the 4 examples, respectively. The X axis has the five risks, and the Y axis has the contribution of each of the risks to the total allocation (in percent).

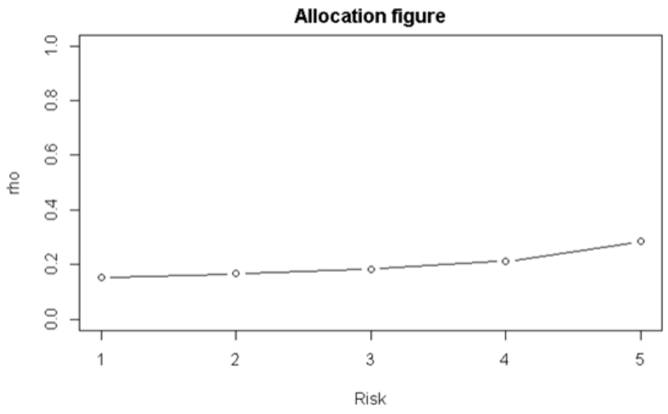

Figure 3.1:

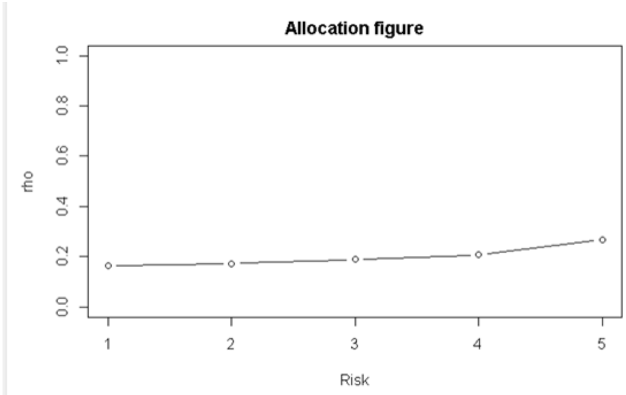

Figure 3.2:

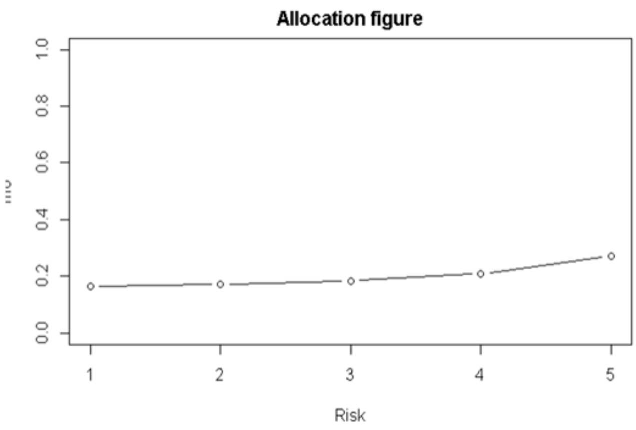

Figure 3.3:

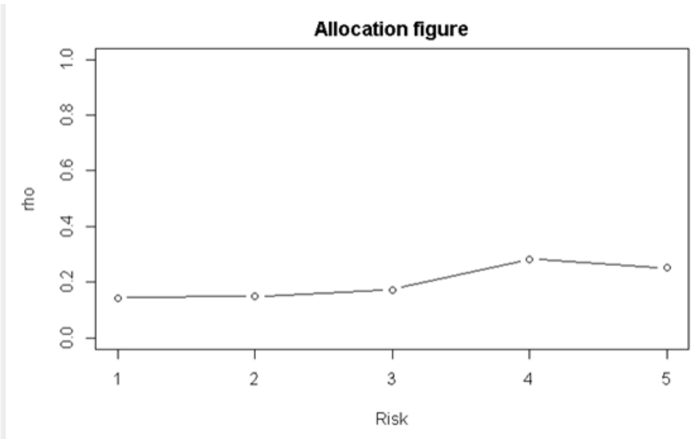

Figure 3.4:

Chapter 4

Multivariate flexible Pareto model

Multivariate Pareto type II distribution has a crucial disadvantage: any univariate marginal distribution has the same shape parameter, which means that they have the same distribution up to a scaling parameter. However, it is difficult to believe that all risk components of some system have the same rate of decrease of the tail distribution for large risks. Moreover, the dependence structure of multivariate Pareto is quite poor, because it allows only equal correlations for each couple of risks. Besides, the independent univariate Pareto marginals do not belong to the multivariate family. In what follows, Arthur Chiragiev and Zinoviy Landsman suggest two new multivariate versions of Pareto distribution, whose univariate marginals are Pareto, but with different shape parameters. They also have a lucratively richer dependence structure, i.e., a flexible one. The first model, which is called Multivariate Flexible Pareto type I (MFP(I)), is the distribution where the power parameters of marginals do not depend on the order of the components included in the model. For the second one, called Multivariate Flexible Pareto type II (MFP(II)), the power parameters are already dependent on the order of their marginals. Therefore, the first model might be considered more attractive; for the second, some important dependence attributes can be calculated in a simpler form. Both models are introduced by the mixture of independent multivariate exponential distributions with respect to their rates.

4.1 Flexible distribution type I

4.1.1 Tail distribution and density functions

Notice that Arnold's distribution (3.1) can be obtained as

$$\overline{F}_{\mathbf{X}}(\mathbf{x}, \boldsymbol{\sigma}, \alpha) = E_G(exp(-\sum_{i=1}^{n} \frac{x_i}{\sigma_i}\lambda)). \quad x_i \geq 0.i = 1, 2, ..., n, \tag{4.1}$$

where $E_G(.)$ is the expectation with respect to the Gamma distributed mixture parameter, such that
$\lambda \sim G(\alpha, 1)$.
Chiragev and Landsman in there article "Multivariate flexible Pareto model, (2009)", weaken the dependency structure of the multivariate mixture parameter
$\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n)$,
used for definition of (4.1). They denote $\lambda_i = Y_0 + Y_i$, where

$$Y_i \sim G(\gamma_i, 1), \qquad i = 0, 1, ..., n. \tag{4.2}$$

Then it follows that
$\lambda_i \sim G(\gamma_0 + \gamma_i, 1), \qquad i = 1, 2, ..., n.$

This approach can be considered as an evaluation of the Marshall and Olkin (1988) principle for the Mathai and Moshopoulos (1991) multivariate gamma (MG) dependence structure.
Construction (4.1)-(4.2) results in an essentially more flexible multivariate Pareto distribution. Chiragev and Landsman call a distribution

$$\overline{F}_{\mathbf{X}}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\alpha}) = E_{MG}(exp(-\sum_{i=1}^{n} \frac{x_i}{\sigma_i}\lambda_i)) \qquad x_i \geq 0, i = 1, 2, ..., n \tag{4.3}$$

multivariate flexible Pareto type I (MFP(I)) distribution, where $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_n)^T$ are the scale parameters and $\alpha_i = \gamma_0 + \gamma_i, \qquad i = 1, 2, ..., n,$
are the shape parameters.

4.2 Flexible distribution type II

In what follows, Chiragiev and Landsman(2009) used another dependence structure, also referred to by Mathai and Moshopoulos (1991).

4.2.1 Tail distribution function

Recall that $Y_1, ..., Y_n$ are independent gamma distributed random variables, as in (4.2). Denote

$$\lambda_i = Y_1 + ... + Y_i, \qquad i \in [1, ..., n]. \tag{4.4}$$

For the case of $\mathbf{X} = (X_1, ..., X_n)$, the MFP(II) tail distribution function is

$$\overline{F}_{\mathbf{X}}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\nu}) = \prod_{i=1}^{n} (1 + \sum_{j=i}^{n} \frac{x_j}{\sigma_j})^{-\gamma_i}, \tag{4.5}$$

and we write $\mathbf{X} \sim MFP(II)_n(\boldsymbol{\sigma}, \boldsymbol{\nu})$, where $\boldsymbol{\nu} = (\nu_1, ..., \nu_n)$, with

$$\nu_i = \sum_{j=1}^{i} \gamma_j, \qquad i = 1, 2, ..., n. \tag{4.6}$$

The MFP(II) density function has the form

$$f_{\mathbf{X}}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\nu}) = \prod_{k=1}^{n} \frac{1}{\Gamma(\nu_k - \nu_{k-1})\sigma_k} (1 + \frac{x_k}{\sigma_k} + .. + \frac{x_n}{\sigma_n})^{-(\nu_k - \nu_{k-1})} \sum_{i_j \in I} a(i_1, .., i_n) \prod_{j=1}^{n} \frac{\Gamma(\nu_j - \nu_{j-1} + i_j)}{(1 + \frac{x_j}{\sigma_j} + .. + \frac{x_n}{\sigma_n})^{i_j}} \tag{4.7}$$

where $a(i_1, ..., i_n), i_j$, and I is considered by

$$y_1(y_1 + y_2)...(y_1 + ... + y_n) = \sum_{i_j \in I} a(i_1, ..., i_n) \prod_{j=1}^{n} y_j^{i_j}, \tag{4.8}$$

where I is some set of indexes i_j, j=1,2,...,n, such that

$$\sum_{j=1}^{n} i_j = n,$$

and $a(i_1, ..., i_n), i_j \in I$, are the appropriate constants.

The expectation of X_i, i=1,2,...,n, is as follows

$$E(X_i) = \frac{\sigma_i}{\nu_i - 1}, \qquad \nu_i \geq 1,$$

and the variance of X_i can be derived as

$$V(X_i) = \frac{\nu_i \sigma_i^2}{(\nu_i - 1)^2(\nu_i - 2)}, \qquad \nu_i > 2, i = 1, 2, ..., n,$$

where ν_i is as in (4.6). The covariance between X_i and X_k, 1≤ i ≤ k ≤ n, has the form

$$Cov(X_i, X_k) = \frac{\sigma_i \sigma_k}{(\nu_i - 1)(\nu_k - 1)(\nu_k - 2)}, \qquad \nu_i > 1, \nu_k > 2.$$

Thus, the correlation coefficient is

$$\rho_{X_i,X_k} = \sqrt{\frac{(\nu_i - 2)}{\nu_i \nu_k(\nu_k - 2)}}, \qquad \nu_i, \nu_k > 2.$$

4.3 MTCE of bivariate flexible Pareto type II

Consider the bivariate flexible Pareto type II distribution

$$\mathbf{X} = (X_1, X_2) \sim MFP(II)_2(\boldsymbol{\sigma}, \boldsymbol{\nu}),$$

where

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2) \quad and \quad \boldsymbol{\nu} = (\nu_1, \nu_2)$$

The MTCE is defined by

$$MTCE_{\mathbf{q}}(\mathbf{X}) = E(\mathbf{X}|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) = \frac{\int_{\boldsymbol{x_q}}^{\infty} \boldsymbol{x} f_{\mathbf{X}}(\boldsymbol{x}) d\boldsymbol{x}}{\overline{F}_{\mathbf{X}}(\boldsymbol{x_q})}.$$

Then,

$$E(X_1|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) =$$

$$\int_{y_2=0}^{\infty} \int_{y_1=0}^{\infty} \prod_{i=1}^{2} f_{Y_i}(y_i) \int_{X_2=VaR_{q_2}(X_2)}^{\infty} \int_{X_1=VaR_{q_1}(X_1)}^{\infty} x_1 \prod_{i=1}^{2} f_{X_i|Y_2,Y_1}(x_i|Y_2 = y_2, Y_1 = y_1) dx_1 dx_2 dy_1 dy_2.$$

By use integration by parts, we get

$$E(X_1|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) = \int_{y_2=0}^{\infty} \int_{y_1=0}^{\infty} \prod_{i=1}^{2} f_{Y_i}(y_i)$$
$$\times \int_{X_2=VaR_{q_2}(X_2)}^{\infty} f_{X_2|Y_2,Y_1}(x_2|Y_2 = y_2, Y_1 = y_1)$$
$$\times([-x_1 e^{\frac{-\lambda_1}{\sigma_1} x_1}]_{VaR_{q_1}(X_1)}^{\infty} - \int_{VaR_{q_1}(X_1)}^{\infty} -e^{\frac{-\lambda_1}{\sigma_1} x_1} dx_1) dx_2 dy_1 dy_2$$

$$= \int_{y_2=0}^{\infty} \int_{y_1=0}^{\infty} \prod_{i=1}^{2} f_{Y_i}(y_i)$$

$$\times \int_{X_2=VaR_{q_2}(X_2)}^{\infty} (VaR_{q_1}(X_1) e^{\frac{-\lambda_1 VaR_{q_1}(X_1)}{\sigma_1}} - [\frac{\sigma_1}{\lambda_1} e^{\frac{-\lambda_1 x_1}{\sigma_1}}]_{VaR_{q_1}(X_1)}^{\infty})$$

$$\times f_{X_2|Y_2,Y_1}(x_2|Y_2 = y_2, Y_1 = y_1) dx_2 dy_1 dy_2$$

$$= \int_{y_2=0}^{\infty} \int_{y_1=0}^{\infty} \prod_{i=1}^{2} f_{Y_i}(y_i)$$

$$\times \int_{X_2=VaR_{q_2}(X_2)}^{\infty} (VaR_{q_1}(X_1) e^{\frac{-\lambda_1}{\sigma_1} VaR_{q_1}(X_1)} + \frac{\sigma_1}{\lambda_1} e^{\frac{-\lambda_1}{\sigma_1} VaR_{q_1}(X_1)}) \frac{\lambda_2}{\sigma_2} e^{\frac{-\lambda_2}{\sigma_2} x_2} dx_2 dy_1 dy_2$$

$$= \int_{y_2=0}^{\infty} \int_{y_1=0}^{\infty} \prod_{i=1}^{2} f_{Y_i}(y_i)$$

$$\times (VaR_{q_1}(X_1) e^{\frac{-\lambda_1}{\sigma_1} VaR_{q_1}(X_1)} + \frac{\sigma_1}{\lambda_1} e^{\frac{-\lambda_1}{\sigma_1} VaR_{q_1}(X_1)}) e^{\frac{-\lambda_2}{\sigma_2} VaR_{q_2}(X_2)} dy_1 dy_2$$

by using eq.(4.2), where

$$\lambda_i = Y_1 + ... + Y_i, \qquad i \in [1, ..., n],$$

we get

$$E(X_1|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X}))$$

$$= \int_{y_2=0}^{\infty} f_{Y_2}(y_2) e^{-\frac{VaR_{q_2}(X_2)}{\sigma_2} y_2} \int_{y_1=0}^{\infty} \frac{y_1^{\gamma_1 - 1} e^{-y_1}}{\Gamma(\gamma_1)} e^{-(\frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{VaR_{q_2}(X_2)}{\sigma_2}) y_1} (VaR_{q_1}(X_1) + \frac{\sigma_1}{y_1}) dy_1 dy_2,$$

we will use the complement to Gamma distribution.

Denote,

$$l = (1 + \frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{VaR_{q_2}(X_2)}{\sigma_2}).$$

So,

$$E(X_1|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) = \int_{y_2=0}^{\infty} f_{Y_2}(y_2) e^{-\frac{y_2 VaR_{q_2}(X_2)}{\sigma_2}} (\frac{VaR_{q_1}(X_1)}{l^{\gamma_1}} + \frac{\sigma_1 \Gamma(\gamma_1 - 1)}{\Gamma(\gamma_1) l^{\gamma_1 - 1}}) dy_2$$

$$= \int_{y_2=0}^{\infty} f_{Y_2}(y_2) e^{-\frac{y_2 VaR_{q_2}(X_2)}{\sigma_2}} (\frac{VaR_{q_1}(X_1)}{l^{\gamma_1}} + \frac{\sigma_1}{(\gamma_1-1)l^{\gamma_1-1}}) dy_2$$

$$= \int_{y_2=0}^{\infty} \frac{y_2^{\gamma_2-1} e^{-(1+\frac{VaR_{q_2}(X_2)}{\sigma_2})y_2}}{\Gamma(\gamma_2)} (\frac{VaR_{q_1}(X_1)}{l^{\gamma_1}} + \frac{\sigma_1}{(\gamma_1-1)l^{\gamma_1-1}}) dy_2,$$

after using a Gamma complement we got

$$E(X_1|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) = \frac{VaR_{q_1}(X_1)}{l^{\gamma_1}(1+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{\gamma_2}} + \frac{\sigma_1}{(\gamma_1-1)l^{\gamma_1-1}(1+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{\gamma_2}}.$$

Denote,

$$l* = \frac{VaR_{q_1}(X_1)}{l^{\gamma_1}(1+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{\gamma_2}} + \frac{\sigma_1}{(\gamma_1-1)l^{\gamma_1-1}(1+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{\gamma_2}}.$$

Then, by using eq.(4.5) we get

$$E(X_1|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) = \frac{l*}{\overline{F}_{\mathbf{X}}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\nu})} = VaR_{q_1}(X_1) + \frac{\sigma_1}{\gamma_1-1}(1+\frac{VaR_{q_1}(X_1)}{\sigma_1} + \frac{VaR_{q_2}(X_2)}{\sigma_2}).$$

Now, we will express the second component of the MTCE vector.

$$E(X_2|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) = \int_{y_2=0}^{\infty} \int_{y_1=0}^{\infty} \prod_{i=1}^{2} f_{Y_i}(y_i)$$

$$\times \int_{X_1=VaR_{q_1}(X_1)}^{\infty} \int_{X_2=VaR_{q_2}(X_2)}^{\infty} x_2 \prod_{i=1}^{2} f_{X_i|Y_2,Y_1}(x_i|Y_2=y_2, Y_1=y_1) dx_2 dx_1 dy_1 dy_2.$$

By using integration by parts, we get

$$E(X_2|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) = \int_{y_2=0}^{\infty} \int_{y_1=0}^{\infty} \prod_{i=1}^{2} f_{Y_i}(y_i)$$

$$\times \int_{X_2=VaR_{q_2}(X_2)}^{\infty} f_{X_1|Y_2,Y_1}(x_1|Y_2=y_2, Y_1=y_1)$$

$$\times([-x_2 e^{\frac{-\lambda_2}{\sigma_2}x_2}]_{VaR_{q_2}(X_2)}^{\infty} - \int_{VaR_{q_2}(X_2)}^{\infty} -e^{\frac{-\lambda_2}{\sigma_2}x_2} dx_2) dx_1 dy_1 dy_2$$

$$= \int_{y_2=0}^{\infty} \int_{y_1=0}^{\infty} \prod_{i=1}^{2} f_{Y_i}(y_i)$$

$$\times \int_{X_1=VaR_{q_1}(X_1)}^{\infty} (VaR_{q_2}(X_2)e^{\frac{-\lambda_2}{\sigma_2}VaR_{q_2}(X_2)} + \frac{\sigma_2}{\lambda_2}e^{\frac{-\lambda_2}{\sigma_2}VaR_{q_2}(X_2)})\frac{\lambda_1}{\sigma_1}e^{\frac{-\lambda_1}{\sigma_1}x_1}dx_1dy_1dy_2$$

$$= \int_{y_2=0}^{\infty} \int_{y_1=0}^{\infty} \prod_{i=1}^{2} f_{Y_i}(y_i)(VaR_{q_2}(X_2)e^{\frac{-\lambda_2}{\sigma_2}VaR_{q_2}(X_2)} + \frac{\sigma_2}{\lambda_2}e^{\frac{-\lambda_2}{\sigma_2}VaR_{q_2}(X_2)})e^{\frac{-\lambda_1}{\sigma_1}VaR_{q_1}(X_1)}dy_1dy_2.$$

Denote,

$$I_1 = \int_{y_2=0}^{\infty} \int_{y_1=0}^{\infty} \prod_{i=1}^{2} f_{Y_i}(y_i)(VaR_{q_2}(X_2)e^{-(\frac{\lambda_2}{\sigma_2}VaR_{q_2}(X_2) + \frac{\lambda_1}{\sigma_1}VaR_{q_1}(X_1))dy_1dy_2$$

$$I_2 = \int_{y_2=0}^{\infty} \int_{y_1=0}^{\infty} \frac{\sigma_2}{\lambda_2} \prod_{i=1}^{2} f_{Y_i}(y_i)e^{-(\frac{\lambda_2}{\sigma_2}VaR_{q_2}(X_2) + \frac{\lambda_1}{\sigma_1}VaR_{q_1}(X_1))dy_1dy_2.$$

By using eq.(4.2), where

$$\lambda_i = Y_1 + ... + Y_i, \qquad i \in [1, ..., n].$$

We get

$$I_1 = \int_{y_2=0}^{\infty} f_{Y_2}(y_2)e^{-\frac{VaR_{q_2}(X_2)}{\sigma_2}y_2} \int_{y_1=0}^{\infty} \frac{y_1^{\gamma_1-1}e^{-y_1}}{\Gamma(\gamma_1)}e^{-(\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})y_1}VaR_{q_2}(X_2)dy_1dy_2.$$

$$I_2 = \int_{y_2=0}^{\infty} f_{Y_2}(y_2)e^{-\frac{VaR_{q_2}(X_2)}{\sigma_2}y_2} \int_{y_1=0}^{\infty} \frac{y_1^{\gamma_1-1}e^{-y_1}}{\Gamma(\gamma_1)}e^{-(\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})y_1}\frac{\sigma_2}{y_1+y_2}dy_1dy_2.$$

By using the complement to Gamma distribution, we get,

$$I_1 = \int_{y_2=0}^{\infty} f_{Y_2}(y_2)e^{-\frac{VaR_{q_2}(X_2)}{\sigma_2}y_2}dy_2 \int_{y_1=0}^{\infty} \frac{y_1^{\gamma_1-1}e^{-y_1}}{\Gamma(\gamma_1)}e^{-(\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})y_1}VaR_{q_2}(X_2)dy_1$$

$$= \int_{y_2=0}^{\infty} f_{Y_2}(y_2)e^{-\frac{VaR_{q_2}(X_2)}{\sigma_2}y_2}\frac{VaR_{q_2}(X_2)}{(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{\gamma_1}}dy_2$$

$$= \frac{VaR_{q_2}(X_2)}{(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{\gamma_1}(1+\frac{VaR_{q_2}(X_2)}{\sigma_2})^{\gamma_2}}.$$

In addition, we used Maple program in order to solve the I_2 integral,

$$I_2 = \int_{y_2=0}^{\infty} f_{Y_2}(y_2)e^{-\frac{VaR_{q_2}(X_2)}{\sigma_2}y_2}dy_2 \int_{y_1=0}^{\infty} \frac{y_1^{\gamma_1-1}e^{-y_1}}{\Gamma\gamma_1}e^{-(\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})y_1}\frac{\sigma_2}{y_1+y_2}dy_1$$

$$= \int_{y_2=0}^{\infty} f_{Y_2}(y_2)e^{-\frac{VaR_{q_2}(X_2)}{\sigma_2}y_2}dy_2$$

$$\times y_2^{\gamma_1-1}\Gamma(\gamma_1)e^{y_2(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})}\Gamma(-\gamma_1+1, y_2(1+\frac{VaR_{q_1}(X_1)}{\sigma_1}+\frac{VaR_{q_2}(X_2)}{\sigma_2})).$$

The last integral we can not success to solve it by the different integral‘s methods, it will be solved numerically in a future research.

Finally, we get that

$$E(X_2|\mathbf{X} > VaR_{\mathbf{q}}(\mathbf{X})) = VaR_{q_2}(X_2) + \frac{I_2}{\overline{F}_{\mathbf{X}}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\nu})}.$$

Chapter 5

Conclusion

In this research we developed expressions for tail conditional expectation (TCE) and tail variance (TV) for univariate Pareto type II distribution. After that, we gave the definition and explanation of properties for multivariate tail conditional expectation (MTCE) and developed expression for it.
Also we developed expression for the capital allocation based on it, in the case of a multivariate Pareto type II distribution.
In addition, we developed expressions for multivariate tail covariance matrix and multivariate tail correlation matrix (MTCov and MTCorr respectively). Furthermore, we get that the dependence structure of multivariate Pareto is quite poor, because it allows only equal correlations for each couple of risks. This unrealistic result drew us to express the same risk measures: MTCE, MTCov, MTCorr, for another distribution called "Multivariate Flexible Pareto", in order to see if a different result would be obtained, that is, whether the correlation is different between any two risks. Multivariate Flexible Pareto type II (MFP(II)), the power parameters are already dependent on the order of their marginals. In our thesis we did not success to got these risk measures.
For future research, we suggest deriving these risk measures for multivariate flexible Pareto distribution, introduced by Chiragiev and Landsman (2009).

Bibliography

[1] A. A. Balkema and L. de Haan(2018). Residual Life Time at Great Age.The Annals of Probability, Vol. 2, No. 5 (Oct., 1974), pp. 792-804.

[2] Alai, D., Landsman, Z, and Sherris, M. (2016). Modelling lifetime dependence for older ages using a multivariate Pareto distribution. *Elsevier journal,***70,** 272-285.

[3] Cai, J. and Li, H. (2005). Conditional tail expectations for multivariate phase type distributions. *Journal of Applied Probability* **42**, 810-825.

[4] Cai, J. and Tan, K.S. (2005). CTE and capital allocation under the skew elliptical distributions. *Working Paper*.

[5] Chiragiev, A., and Landsman, Z. (2009). "Multivariate flexible Pareto model: Dependency structure, properties and characterizations." *Statistics & Probability Letters*, **79,** 1733-1743.

[6] Cebrien, A., Denuit, M,and Lambert, P (2003). Generalized Pareto Fit to the Society of Actuaries Large Claims Database. *North American Actuarial Journal*, *7:3*, 18-36.

[7] Devolder, P., and Lebegue, A. (2016). Compositions of Conditional Risk Measures and Solvency Capital. Institut de Statistique, Biostatistique et Sciences Actuarielles, Universite© catholique de Louvain, Voie du Roman Pays 20 bte L1.04.01, B-1348 Louvain-la-Neuve, Belgium.

[8] Fang, K. T., Kotz, S. and Ng, K. W. (1987). "Symmetric multivariate and related distributions". *London: Chapman and Hall.*

[9] Furman, E. and Landsman, Z. Furman, E. and Landsman, Z. (2005). Risk Capital Decomposition for a Multivariate Dependent Gamma Portfolio. *Insurance: Mathematics and Economics* **37**, 635-649.

[10] Landsman, Z. and Valdez, E.A. (2003). Tail Conditional Expectation for Elliptical Distributions. *North American Actuarial Journal* **7**, 55-71.

[11] Landsman, Z. and Valdez, E.A. (2005). Tail Conditional Expectation for Exponential Dispersion Models. *Astin Bulletin* **35**, 189-209.

[12] Landsman, Z., Makov, U., and Shushi, T. (2016). Multivariate tail conditional expectation for elliptical distributions. *Insurance: Mathematics and Economics*, **70**, 216-223.

[13] Landsman, Z., Makov, U., and Shushi, T. (2018). A multivariate tail covariance measure for elliptical distribution. *Insurance: Mathematics and Economics*, **81**, 27-35

[14] Li Zhu, and Haijun Li. (2012). Asymptotic Analysis of Multivariate Tail Conditional Expectations. *North American Actuarial Journal.***16:3,** 350-363.

[15] Mardia, K.V. (1962). Multivariate Pareto Distributions. *The Annals of Mathematical Statistics* **33**, 1008-1015.

[16] McNeil A., Frey, R. and Embrechts, P. (2005). *Quantitative Risk Management: Concepts Techniques Tools.* Princeton University Press, Princeton Series in Finance.

[17] Panjer, H.H. (2002). Solvency and Capital Allocation. *Institute of Insurance and Pension Research, University of Waterloo, Research Report*, 01-14.

[18] Pickands, J. (1975). Statistical Inference Using Extrime Order Statistics. *The Annals of Statistics* **3**, 119-131.

[19] Renyi, A. (1970). *Probability Theory.* North-Holland, Amsterdam.

[20] Tasche, D. (1999). Risk Contributions and Performance Measurement. *Working Paper, Technische Universität München.*

[21] Vernic, R. (2006). Multivariate skew-normal distributions with applications in insurance. *Insurance: Mathematics and Economics* **38**, 413-426.

[22] Volkov, E.A. (1986). *Numerical Methods.* Mir Publishers, Moscow.

[23] Wang, S. (1996). Premium Calculation by Transforming the Layer Premium Density. *Astin Bulletin* **26**, 71-92.

[24] Wang, S. (1997). Implementation of PH-Transforms in Ratemaking. *Proccedings of the Casualty Actuarial Society,* Vol **LXXV**, 940-979.

[25] Wang, S. (1998). An Actuarial Index of the Right-Tail Risk. *North American Actuarial Journal* **2**, 88-101.

[26] Wang, S., Young, V.R., and Panjer, H.H. (1997). Axiomatic Characterization of Insurance Prices. *Insurance: Mathematics and Economics* **21**, 173-183.

[27] Wirch, J. and Hardy, M. (1999). A Synthesis of Risk Measures for Capital Adequacy. *Insurance: Mathematics and Economics* **25**, 337-348.

[28] Yeh, H.C. (2004). Some Properties and Characterizations for Generalized Multivariate Pareto Distribution. *Journal of Multivariate Analysis* **88**, 47-60.

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תקציר

חברות ביטוח שומרות רזרבות כדי להבטיח את יכולתן לעמוד בכל התביעות העתידיות. את קביעת גודל הרזרבה מהווה עניין עיקרי מבחינתן. כלומר, הן צריכות להעריך את סכום ההפסד שיש להן בתקופת זמן נתונה. בזמן האחרון, חקירת הזנב של התפלגות מסוימת שמתארת את גודל התביעה מהווה עניין חשוב בקרב חברות ביטוח וגופים פיננסיים אחרים. הכוונה בחקירת הזנב היא להשתמש במדדי סיכון על מנת להעריך את ההפסד שהחברה עלולה להיחשף אליו בתקופה נתונה, בהינתן שהסיכון גדול מרף מסוים. ערכו של רף זה מבוסס על אחוזון של התפלגות ההפסד והינו מכונה Value at Risk (VaR).

לאור הנאמר, במסגרת התזה בחרנו למצוא ביטויים עבור מדדי הסיכון השונים עבור התפלגות פרטו רב ממדית מסוג II שהיא די חשובה בעבודה המעשית של אקטוארים. בהתחלה מצאנו ביטוי עבור תוחלת מותנית של זנב ההתפלגות, תוחלת זו מכונה TCE (Tail Conditional Expectation), עבור משתנה סיכון המסומן ב- X, היא מוגדרת על ידי

$$TCE_q(X) = E(X|X > x_q).$$

מידת סיכון זו יכולה להיות מפורשת כהפסד מרבי, כאשר היא נותנת הגודל הממוצע של זנב ההתפלגות, הזנב הוא מעבר לאחוזון ספציפי של התפלגות ההפסד, כאשר הערך של פונקציית השרידות בנקודה x_q שווה ל- 1-q, כאשר 0<q<1.

למדד הנ״ל יש כמה חסרונות ולכן פותח מדד מוכלל יותר הנקרא MTCE (Multivariate Tail Conditional Expectation), אשר הוצג על ידי לנדסמן (2016). מדד זה לוקח בחשבון את התלות בין המשתנים השונים, אשר הוא המקרה כאשר אנו עוסקים בנתונים אמתיים של הפסדים.

במסגרת התזה מצאנו ביטוי של MTCE עבור וקטור המורכב מ- n סיכונים המתפלגים פרטו, אבל הכללנו את ההגדרה של MTCE להיות עבור -n אחוזונים שונים. כלומר, לכל אחד מהסיכונים בחרנו רמת ביטחון שונה q_i כאשר i=1,…,n . ה- MTCE מוגדר על ידי

$$MTCE_{\mathbf{q}}(\mathbf{X}) = E\left(\mathbf{X}\middle|X > VaR_{\mathbf{q}}(X)\right) = E(\mathbf{X}|X_1 > VaR_{q_1}(X_1), \dots, X_n > VaR_{q_n}(X_n)),$$

$$0 < q_i < 1, i = 1, \dots, n.$$

בנוסף לכך, מצאנו ביטויים עבור מטריצת השונויות המותנות של זנב ההתפלגות המכונה MTCov (Multivariate Tail Covariance matrix). השתמשנו במטריצת השונויות שהתקבלה בכדי לבטא את מטריצת הקורלציה (Multivariate Tail Correlation matrix) MTCorr. במטריצת הקורלציה קיבלנו שמקדם המתאם שווה בין כל זוג סיכונים, תוצאה שנראית מוזרה ולא מציאותית.

התוצאות שהתקבלו עבור התפלגות פרטו גרמו לנו לחשוב שצריך לחפש התפלגות אחרת שיכולה להתאים יותר לתיאור התנהגות הסיכונים. בחרנו בהתפלגות שנקראת Multivariate Flexible Pareto type II, ניסינו למצוא ביטויים למדדי הסיכון הנ״ל אבל לא הצלחנו במסגרת התזה וכנראה הפתרון הוא נומרי.